Vladikavkaz Mathematical Journal
2026, Volume 28, Issue 2, P. 5–19

УДК 517.98, 512.643.8
DOI 10.46698/b6298-9094-1158-n

CONDITIONAL BASES IN THE MATRIX SPACE WITH THE CUT-NORM[#]

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Dedicated to the 85th anniversary of E. M. Semenov

Abstract. In the paper we consider the space $\mathscr{L}_{0,1}$ of infinite matrices $(a_{ij})_{i,j=1}^{\infty}$, where $a_{ij} \in \mathbb{R}$, equipped with a cut-norm. The cut-norm of a matrix appears naturally in various areas of discrete mathematics, and is also used in functional analysis. It is equivalent to the operator norm of a matrix considered as a mapping from the space c_0 of sequences converging to zero (or from l^∞) to l^1. It is proved that the set of matrix units $\{E_{ij}\}_{i,j=1}^{\infty}$ forms a system of random unconditional convergence (RUC) in the space $\mathscr{L}_{0,1}$, but is not a system of unconditional convergence. This system is linearly full in $\mathscr{L}_{0,1}$ and, being enumerated in a certain order, may turn out to be a conditional basis. We obtain both necessary and sufficient conditions for a bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$, under which the sequence $(E_{\ell(k)})$ forms a basis in $\mathscr{L}_{0,1}$. In particular, any successive filling of the corner squares leads to a basis, while the triangular filling not. It is shown that random permutations of the sequence $(E_{\ell(k)})$ almost surely do not form a basis. The problem of a complete characterization of all bijections $\ell\colon \mathbb{N} \to \mathbb{N}^2$ that lead to the basis $(E_{\ell(k)})$ is set up, and it is constructively shown that the proposed necessary and sufficient conditions do not provide a complete answer to this problem.

Keywords: conditional basis, RUC system, infinite matrix space, operator norm, cut-norm.

AMS Subject Classification: 46B15, 46B28, 46B45.

For citation: *Astashkin, S. V., Bakhtin, V. I. and Lykov, K. V.* Conditional Bases in the Matrix Space with the Cut-Norm, *Vladikavkaz Math. J.*, 2026, vol. 28, no. 2, pp. 5–19. DOI: 10.46698/b6298-9094-1158-n.

1. Introduction and Preliminaries

In what follows, by a *basis* in a Banach space X we mean the Schauder basis, that is, a sequence of elements $(e_k)_{k=1}^{\infty}$ of X, such that for any vector $x \in X$ there exists a unique expansion $x = \sum_{k=1}^{\infty} a_k e_k$. In this paper we will consider only real Banach spaces and scalars $a_k \in \mathbb{R}$.

[#] The work of the first author was completed as a part of the implementation of the development program of the Volga Region Scientific and Educational Mathematical Center (agreement no. 075-02-2025-1791). The second author was supported by the State Research Program "Convergence-2030", Project no. 1.08.3. The third author was supported by the State Research Program "Convergence-2025", Project no. 1.3.05, and by the State Research Program "Convergence-2030", Project no. 1.08.1.

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Recall that a basis $(e_k)_{k=1}^{\infty}$ in a Banach space X is called *unconditional* if for any permutation π of the set of positive integers the sequence $(e_{\pi(k)})_{k=1}^{\infty}$ is a basis as well. Otherwise, the basis is called *conditional*. It is well known that a basis $(e_k)_{k=1}^{\infty}$ is unconditional if and only if there exists a constant $D > 0$, such that for every $n \in \mathbb{N}$, any signs $\theta_k = \pm 1$, and any coefficients $a_k \in \mathbb{R}$,

$$\left\| \sum_{k=1}^{n} \theta_k a_k e_k \right\|_X \leqslant D \left\| \sum_{k=1}^{n} a_k e_k \right\|_X. \tag{1}$$

A detailed account of properties of bases and unconditional bases can be found, for instance, in the books [1–3].

Relaxing the above definition, we say that a sequence $(e_k)_{k=1}^{\infty}$ is a *system of random unconditional convergence* (in brief, a RUC system) whenever there exists a constant $D' > 0$, such that for any $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$ we have

$$\mathsf{E}_\theta \left\| \sum_{k=1}^{n} \theta_k a_k x_k \right\|_X \leqslant D' \left\| \sum_{k=1}^{n} a_k x_k \right\|_X, \tag{2}$$

where the expectation E_θ is taken over all 2^n equally probable arrangements of the signs $\theta_k = \pm 1$. This concept was introduced by Billard, Kwapień, Pełczyński and Samuel in [4].

There is no reason to believe that the RUC-property of a system entails in any way its being a basis. In particular, there are examples of complete (linearly dense) RUC systems that are not bases for any numberings of their elements [4, Example 3.1]. At the same time, the comparison of (1) and (2) shows that bases with RUC-property are, in a certain sense, close to unconditional ones. In this connection, the RUC systems were studied thoroughly in the series of works [5–10].

Of particular interest is the existence of RUC bases in matrix spaces because, as a rule, they possess no unconditional bases (see, e. g., [11, Theorem 2.3]). In this paper we consider this question for the space of matrices equipped with the cut-norm.

Let L_σ be a linear normed space of real-valued matrices $(a_{ij})_{i,j=1}^{\infty}$ with a norm σ. From now on, let E_{ij} stand for the matrix in which the element a_{ij} is equal to one, and all the other ones are zero. Following [11], we will call σ an *unconditional matrix norm* if:

(a) $E_{ij} \in L_\sigma$ and $\sigma(E_{ij}) = 1$ for all $i, j \in \mathbb{N}$;

(b) for every matrix $A = (a_{ij}) \in L_\sigma$ and sequences $\varepsilon = (\varepsilon_i)$ and $\delta = (\delta_j)$ of signs the matrix $A^{\varepsilon,\delta} = (\varepsilon_i \delta_j a_{ij})$ belongs to L_σ and $\sigma(A^{\varepsilon,\delta}) = \sigma(A)$.

Let us define the *rectangular numbering* of the system of unit matrices $\{E_{ij}\}$ as follows: an element $E_{i_1 j_1}$ precedes $E_{i_2 j_2}$, if either $\max(i_1, j_1) < \max(i_2, j_2)$; or $\max(i_1, j_1) = \max(i_2, j_2)$ and $j_1 < j_2$; or $\max(i_1, j_1) = \max(i_2, j_2) = j_1 = j_2$ and $i_1 < i_2$.

Recall that a Banach space X is of the *Rademacher type* p, where $p \in [1, 2]$, if there exists a constant C_p, such that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$ we have

$$\mathsf{E}_\theta \left\| \sum_{k=1}^{n} \theta_k x_k \right\|_X \leqslant C_p \left(\sum_{k=1}^{n} \|x_k\|_X^p \right)^{1/p}$$

(the expectation E_θ is taken over all 2^n arrangements of the signs $\theta_k = \pm 1$).

The following result shows that the Rademacher type 2 guarantees that the sequence $(E_{ij})_{i,j=1}^{\infty}$ taken in the rectangular order is a RUC basis in the matrix space L_σ.

Theorem 1 [4, Theorem 3.1′]. *Suppose that a space L_σ with an unconditional matrix norm σ has the Rademacher type 2, and let the system $\{E_{ij}\}_{i,j=1}^{\infty}$ be complete (i. e., linearly*

dense) in the space L_σ. Then, the sequence $(E_{ij})_{i,j=1}^\infty$ taken in the rectangular order is a RUC basis in L_σ.

In this paper, we study the Banach space $\mathscr{L}_{0,1} := \mathscr{L}(c_0, l^1)$ of bounded linear operators $A\colon c_0 \to l^1$, where c_0 is the space of all sequences converging to zero with the norm

$$\|x\|_{c_0} := \sup_k |x_k|, \quad x = (x_1, x_2, \ldots),$$

and l^1 is the space of all summable sequences with the norm

$$\|x\|_{l^1} := \sum_{k=1}^\infty |x_k|.$$

In what follows, we identify in a natural way operators $A \in \mathscr{L}_{0,1}$ with matrices $(a_{ij})_{i,j=1}^\infty$. One can easily check that the rows and columns of these matrices belong to the space l^1, and their norms in l^1 do not exceed the norm of A in $\mathscr{L}_{0,1}$.

Observe that some properties of the space $\mathscr{L}_{0,1}$ resemble those of $L^1[0,1]$. In particular, $\mathscr{L}_{0,1}$ is not isomorphic to any subspace of a Banach space with an unconditional basis, and it has the Schur property, that is, in $\mathscr{L}_{0,1}$ weak and strong convergence of sequences coincide [11, Example 2.1]. At the same time, in contrast to $L^1[0,1]$, the space $\mathscr{L}_{0,1}$ is isometrically isomorphic to the dual space of some Banach space [11, Example 2.1].

Here, we will prove one more property of the space $\mathscr{L}_{0,1}$ that distinguishes it from $L^1[0,1]$: it has a RUC basis, while $L^1[0,1]$ has not [4, Corollary 2.2].

Coming back to Theorem 1, note that $\|\cdot\|_{\mathscr{L}_{0,1}}$ possesses the properties (a) and (b) and hence it is an unconditional matrix norm. On the other hand, the space $\mathscr{L}_{0,1}$ fails to have any non-trivial ($p > 1$) Rademacher type. Indeed, in view of [12, Theorem 2] (see also formula (10) there),

$$\mathsf{E}_\theta \left\| \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} a_{ij} E_{ij} \right\|_{\mathscr{L}_{0,1}} \asymp \max \left\{ \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 \right)^{1/2}, \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}^2 \right)^{1/2} \right\},$$

where an expression of the form $F_1 \asymp F_2$ means that $cF_1 \leqslant F_2 \leqslant CF_1$ for some constants $c, C > 0$ independent of n and $a_{ij} \in \mathbb{R}$. Taking $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$, we get

$$\max \left\{ \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 \right)^{1/2}, \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}^2 \right)^{1/2} \right\} = n,$$

while

$$\left(\sum_{i=1}^n \sum_{j=1}^n \|a_{ij} E_{ij}\|_{\mathscr{L}_{0,1}}^p \right)^{1/p} = n^{1/p} \ll n \quad \text{for} \quad p > 1.$$

Thus, Theorem 1 is not applicable to the space $\mathscr{L}_{0,1}$. However, from the same Theorem 2 in [12] it follows that $\{E_{ij}\}_{i,j=1}^\infty$ is a RUC system in this space. We intend to prove here that the sequence $(E_{ij})_{i,j=1}^\infty$ taken in the rectangular order is a RUC basis in $\mathscr{L}_{0,1}$.

More generally, let $\ell\colon \mathbb{N} \to \mathbb{N}^2$ be an arbitrary bijection. It defines a numbering of the system $\{E_{ij}\}$ according to the rule $k \mapsto E_{\ell(k)}$, $k = 1, 2, \ldots$

We are going to study the following question: given a bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$, is $(E_{\ell(k)})_{k=1}^{\infty}$ a basis in the space $\mathscr{L}_{0,1}$? That is,

$$\text{does the equality } \ A = \sum_{k=1}^{\infty} a_{\ell(k)}\, E_{\ell(k)} \ \text{ hold for each } \ A = (a_{ij})_{i,j=1}^{\infty} \in \mathscr{L}_{0,1}? \tag{3}$$

Treating $\mathscr{L}_{0,1}$ as a matrix space, observe that its norm is equivalent to the so-called cut-norm of the matrix $A = (a_{ij})_{i,j=1}^{\infty}$ defined as

$$\|A\|_{\mathrm{cut}} := \sup\left\{ \left| \sum_{i\in I, j\in J} a_{ij} \right| :\ I, J \text{ are finite subsets of } \mathbb{N} \right\}.$$

This norm plays an important role in the study of the efficiency of approximative and computational algorithms (see e.g. [13–16] and references therein). That is why the cut-norm appears in the title of the paper.

2. A Criterion for $(E_{\ell(k)})$ to be a Basis in $\mathscr{L}_{0,1}$

Every continuous linear operator $A\colon c_0 \to l^1$ generates a continuous bilinear functional defined on the space $c_0 \times c_0$ as follows:

$$a(y, x) = \langle y, Ax\rangle, \quad x, y \in c_0, \tag{4}$$

where $\langle x, y\rangle$ denotes the usual scalar product of two sequences $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$. Since the dual space of c_0 coincides isometrically with l^1, the correspondence $a \leftrightarrow A$ is one-to-one and norm-preserving, i. e., $\|a\|_{c_0\times c_0\to\mathbb{R}} = \|A\|_{\mathscr{L}_{0,1}}$.

Further, all subsets of $\mathbb{N}^2$ will be called *masks*. Let (a_{ij}) be the matrix corresponding to an operator $A \in \mathscr{L}_{0,1}$. For each mask $M \subset \mathbb{N}^2$ we set $P_M(A) := B$, where the matrix (operator) $B = (b_{ij})$ is defined by

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } \ (i,j) \in M, \\ 0, & \text{if } \ (i,j) \notin M. \end{cases}$$

The operator P_M is bounded whenever M is finite. Indeed, for each $A = (a_{ij}) \in \mathscr{L}_{0,1}$ we have

$$\|P_M(A)\|_{\mathscr{L}_{0,1}} = \left\| \sum_{(i,j)\in M} a_{ij}\, E_{ij} \right\|_{\mathscr{L}_{0,1}} \leqslant \sum_{(i,j)\in M} |a_{ij}| \leqslant |M| \|A\|_{\mathscr{L}_{0,1}},$$

and hence $\|P_M\| \leqslant |M|$.

Note that each bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$ defines the following sequence of masks:

$$M(k) = \big\{\ell(1), \ell(2), \ldots, \ell(k)\big\}, \quad k \in \mathbb{N}.$$

Since for $A = (a_{ij})$,

$$P_{M(k)}(A) = \sum_{i=1}^{k} a_{\ell(i)}\, E_{\ell(i)},$$

the above question (3) is equivalent to the following one:

$$\text{does the equality } \ \lim_{k\to\infty} P_{M(k)}(A) = A \ \text{ hold for each } \ A \in \mathscr{L}_{0,1}? \tag{5}$$

If the answer to (5) is affirmative, then according to the Banach–Steinhaus principle the set of projections $\{P_{M(k)}\}_{k=1}^{\infty}$ is uniformly bounded. Our first result shows that the converse is also true.

Theorem 2. *The sequence $(E_{\ell(k)})$, corresponding to a bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$, is a basis in the space $\mathscr{L}_{0,1}$ if and only if the set of projections $\{P_{M(k)}\}_{k=1}^{\infty}$ is uniformly bounded.*

According to the well-known criterion (see, e.g., [2, Chapter 1, Theorem 6] or [1, § 1.1]), a system $(x_k)_{k=1}^{\infty}$ is a basis in a Banach space X if and only if this system is minimal, complete, and the canonical projections corresponding to this system are uniformly bounded in X. Clearly, the sequence $(E_{\ell(k)})$ is minimal. Therefore, it only remains to verify that it is complete in the space $\mathscr{L}_{0,1}$. To this end, we will need several lemmas.

For each subset $\alpha \subset \mathbb{N}$, let π_α denote the projection defined on the space l^∞ as follows: if $x = (x_1, x_2, \dots)$, then

$$(\pi_\alpha x)_i = \begin{cases} x_i, & \text{if } \ i \in \alpha, \\ 0, & \text{if } \ i \notin \alpha. \end{cases}$$

One can easily see that $\pi_\alpha \pi_\beta = \pi_{\alpha \cap \beta}$.

Lemma 1. *Let $\alpha = \alpha_1 \sqcup \dots \sqcup \alpha_n$ and $\beta = \beta_1 \sqcup \dots \sqcup \beta_n$, where $\alpha_i, \beta_i \subset \mathbb{N}$. Then for each $A \in \mathscr{L}_{0,1}$ we have*

$$\sum_{i=1}^{n} \|\pi_{\beta_i} A\, \pi_{\alpha_i}\| \leqslant \|\pi_\beta A\, \pi_\alpha\|. \tag{6}$$

$\lhd$ Let $n = 2$. For any $\varepsilon > 0$, by (4), there exist elements $x_1, x_2, y_1, y_2 \in c_0$ with norm equal to one, such that

$$\langle y_i, \pi_{\beta_i} A\, \pi_{\alpha_i} x_i \rangle > \|\pi_{\beta_i} A\, \pi_{\alpha_i}\| - \varepsilon, \quad i = 1, 2.$$

Next, we put

$$u_\pm = \pi_{\alpha_1} x_1 \pm \pi_{\alpha_2} x_2, \quad v_\pm = \pi_{\beta_1} y_1 \pm \pi_{\beta_2} y_2.$$

Since $\alpha_1 \cap \alpha_2 = \beta_1 \cap \beta_2 = \varnothing$, we have that $\|u_\pm\|_{c_0} \leqslant 1$ and $\|v_\pm\|_{c_0} \leqslant 1$. Consequently,

$$2\|\pi_\beta A\, \pi_\alpha\| \geqslant \langle v_+, \pi_\beta A\, \pi_\alpha u_+ \rangle + \langle v_-, \pi_\beta A\, \pi_\alpha u_- \rangle = \langle v_+, A u_+ \rangle + \langle v_-, A u_- \rangle$$
$$= 2\sum_{i=1}^{2} \langle \pi_{\beta_i} y_i, A\, \pi_{\alpha_i} x_i \rangle = 2\sum_{i=1}^{2} \langle y_i, \pi_{\beta_i} A\, \pi_{\alpha_i} x_i \rangle > 2\sum_{i=1}^{2} \|\pi_{\beta_i} A\, \pi_{\alpha_i}\| - 4\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, inequality (6) follows for $n = 2$. Next, we proceed by induction. $\rhd$

Let us define for any $n \in \mathbb{N}$ the projections π_n and σ_n on the space l^∞ by

$$\pi_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots),$$
$$\sigma_n(x_1, x_2, \dots) = (0, \dots, 0, x_{n+1}, x_{n+2}, \dots).$$

Lemma 2. *For every operator $A \in \mathscr{L}_{0,1}$ we have*

$$\lim_{n\to\infty} \|\pi_m A \sigma_n\| = \lim_{n\to\infty} \|\sigma_n A \pi_m\| = 0, \quad m \in \mathbb{N}. \tag{7}$$

$\lhd$ The matrix of the operator $\pi_m A \sigma_n$ is obtained from the matrix $A = (a_{ij})$ by replacing the first n columns and the rows with numbers $i > m$ with zeros. Then, for each fixed m,

$$\|\pi_m A \sigma_n\| \leqslant \sum_{i=1}^{m} \sum_{j=n+1}^{\infty} |a_{ij}| \to 0 \quad \text{as} \ \ n \to \infty.$$

The second equality in (7) can be verified similarly. $\rhd$

Lemma 3. *For every operator $A \in \mathscr{L}_{0,1}$ we have* $\lim_{n\to\infty} \|A\sigma_n\| = \lim_{n\to\infty} \|\sigma_n A\| = 0$.

$\lhd$ Assume the contrary, say, that $\|A\sigma_n\|$ does not tend to zero. Obviously, the sequence $\|A\sigma_n\|$ is nonincreasing. Then there exists $\varepsilon > 0$, such that $\|A\sigma_n\| > \varepsilon$ for all n. Since $\pi_m y + \sigma_m y \equiv y$, by Lemma 2, for each $m \in \mathbb{N}$ it follows $\|\sigma_m A\sigma_n\| > \varepsilon$ whenever n is sufficiently large. On the other hand, the expression $\|\sigma_m A\sigma_n\|$ does not decrease with decreasing n and m. Hence,

$$\|\sigma_m A\sigma_n\| > \varepsilon \quad \text{for all} \quad m, n \in \mathbb{N}. \tag{8}$$

Next, there exist increasing sequences $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ of positive integers that satisfy the condition

$$\|(\sigma_{m_i} - \sigma_{m_{i+1}})A(\sigma_{n_i} - \sigma_{n_{i+1}})\| > \varepsilon. \tag{9}$$

Indeed, assume that the numbers n_i and m_i are already found. Then, by (8), we have $\|\sigma_{m_i} A\sigma_{n_i}\| > \varepsilon$, and hence there is $x \in c_0$ such that $\|x\|_{c_0} = 1$ and $\|\sigma_{m_i} A\sigma_{n_i} x\|_{l^1} > \varepsilon$. Observe that for arbitrary $u \in c_0$ and $v \in l^1$,

$$\lim_{n\to\infty} \|\sigma_n u\|_{c_0} = \lim_{n\to\infty} \|\sigma_n v\|_{l^1} = 0.$$

Consequently, we can find sufficiently large n_{i+1} and m_{i+1}, such that

$$\|(\sigma_{m_i} - \sigma_{m_{i+1}})A(\sigma_{n_i} - \sigma_{n_{i+1}})\, x\|_{l^1} > \varepsilon,$$

which implies (9).

Finally, denoting $\alpha_i = \{n : \ n_i < n \leqslant n_{i+1}\}$ and $\beta_i = \{m : \ m_i < m \leqslant m_{i+1}\}$, we see that inequality (9) may be rewritten in the form:

$$\|\pi_{\beta_i} A\, \pi_{\alpha_i}\| > \varepsilon, \quad i = 1, 2, \dots$$

Since the sets $\alpha_1, \alpha_2, \dots$ (resp. $\beta_1, \beta_2, \dots$) are pairwise disjoint, the last inequality and Lemma 1 imply that A is unbounded, which contradicts the assumption. $\rhd$

As a consequence we get the following result that should be compared with Lemma 1 from the paper [17].

Corollary 1. *Every operator $A \in \mathscr{L}_{0,1}$ can be extended to a continuous linear operator $\tilde{A}\colon l^\infty \to l^1$ by setting $\tilde{A}x := \lim_{n\to\infty} A\pi_n x$, $x \in l^\infty$.*

Now we are ready to give a proof of Theorem 2. As was said above, it suffices to check that the sequence $(E_{\ell(k)})$ is complete in the space $\mathscr{L}_{0,1}$.

From Lemma 3 it follows that, for every operator $A \in \mathscr{L}_{0,1}$ and each $\varepsilon > 0$, there exists a positive integer n, such that

$$\|\pi_n A\sigma_n\| < \varepsilon, \quad \|\sigma_n A\, \pi_n\| < \varepsilon, \quad \|\sigma_n A\sigma_n\| < \varepsilon.$$

Then, we have

$$\|A - \pi_n A\, \pi_n\| \leqslant \|\sigma_n A\, \pi_n\| + \|\pi_n A\, \sigma_n\| + \|\sigma_n A\, \sigma_n\| \leqslant 3\varepsilon.$$

Since the operators (matrices) $\pi_n A\, \pi_n$ are linear combinations of the matrices E_{ij}, and $\varepsilon > 0$ is arbitrary, the proof of Theorem 2 is completed.

Although Theorem 2 provides a criterion for the sequence $(E_{\ell(k)})$ to form a basis in the space $\mathscr{L}_{0,1}$, it is not clear, in general, how to verify the uniform boundedness of the set of projections $\{P_{M(k)}\}_{k=1}^{\infty}$. Hence, it is very desirable to find some geometric conditions, based on the shape of masks $M(k)$, $k = 1, 2, \dots$, that would allow us to determine whether the projections $P_{M(k)}$ are uniformly bounded or not. Some conditions of this type will be given in the next two sections.

3. Uniform Boundedness of $\{P_{M(k)}\}$: a Sufficient Condition

Set $M_{ij} = \{1, \dots, i\} \times \{1, \dots, j\}$ for any $i, j \in \mathbb{N}$. Following [11] (see also [18]), let us define the sequence of masks

$$M(m^2 + s) := \begin{cases} M_{m,m} \cup M_{m+1,s} & \text{for } \ s = 1, \dots, m, \\ M_{m+1,m} \cup M_{s-m,m+1} & \text{for } \ s = m+1, \dots, 2m+1, \end{cases}$$

where $m = 0, 1, 2, \dots$ It is easily seen that these masks are associated with a certain bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$, such that $M(k) = \{\ell(1), \ell(2), \dots, \ell(k)\}$, and each $\ell(k)$ is the unique element of $M(k) \setminus M(k-1)$. Moreover, this bijection defines the rectangular ordering of $\mathbb{N}^2$. In [11, Theorem 2.1] it is proved that, for each unconditional matrix norm σ (see the definition in the Introduction), the sequence $(E_{\ell(k)})$ is a basis in the Banach space L_σ.

By definition, each of the above masks $M(k)$ is just a union of two rectangles in $\mathbb{N}^2$. In this section we introduce more general conditions on subsets of $\mathbb{N}^2$ of such a sort, under which a sequence of masks $M(k) = \{\ell(1), \ell(2), \dots, \ell(k)\}$, $k \in \mathbb{N}$, generates a basis in the matrix space $\mathscr{L}_{0,1}$.

Now let us introduce a couple of new definitions. A subset $M \subset \mathbb{N}^2$ will be called *primitive* if it can be represented in the form:

$$M = \bigcup_{i=1}^{n} \beta_i \times \alpha_i,$$

where $1 \leqslant n \leqslant \infty$, and the sets of positive integers $\alpha_i,\ i = 1, 2, \dots$ (resp. $\beta_i,\ i = 1, 2, \dots$) are pairwise disjoint.

We will say that any algebra of subsets of $\mathbb{N}^2$, generated by arbitrary r primitive sets and their complements, has *rank* r. Clearly, every algebra of rank r has at most 2^r atoms. The rank of a set $M \subset \mathbb{N}^2$ will be understood as the minimum of ranks of algebras containing M. If a set M belongs to no algebra of finite rank, then we say that it has infinite rank. One can easily check that the union, intersection, and difference of any two sets of ranks r and r' respectively have rank at most $r + r'$. In particular, the rank of any union of r primitive sets does not exceed r.

Lemma 4. *If a set $M \subset \mathbb{N}^2$ contains in each row (resp. each column) no more than r elements, then its rank is at most r.*

$\lhd$ If a set M has at most one element in each column, then it is primitive. Indeed, we can take $\beta_i = \{i\}$ and $\alpha_i = \{\, j : (i, j) \in M \,\}$. Then, the sets α_i are pairwise disjoint and $M = \bigcup_i \beta_i \times \alpha_i$.

In the general case, we proceed by induction. If a set M has at most $r+1$ elements in each column, then it can be represented as a union $M_r \cup M_1$, where M_r has at most r elements in each column, and M_1 is primitive. $\rhd$

Lemma 5. *For any set $M \subset \mathbb{N}^2$ of rank r we have $\|P_M\| \leqslant 4^r$.*

$\lhd$ Since every algebra of rank r has at most 2^r atoms, it suffices to prove that $\|P_M\| \leqslant 2^r$ for any atom M of rank r. If M is a primitive set, then Lemma 1 implies the inequality $\|P_M\| \leqslant 1$. In turn, for its complement $\mathbb{N}^2 \setminus M$ we have $\left\|P_{\mathbb{N}^2 \setminus M}\right\| = \|I - P_M\| \leqslant 2$. As a result, the estimate $\|P_M\| \leqslant 2$ holds for each first-rank atom. Next, we take the induction step. Every atom M of rank $r+1$ is an intersection of an atom N_r of rank r and a first-rank atom N_1. Therefore, $\|P_M\| = \|P_{N_1} P_{N_r}\| \leqslant 2 \cdot 2^r = 2^{r+1}$, and so the proof is completed. $\rhd$

Now, we are ready to get a sufficient condition for a sequence of masks to generate the sequence $(E_{\ell(k)})$ being a RUC basis in the space $\mathscr{L}_{0,1}$.

Theorem 3. *If the ranks of all masks $M(k) = \{\ell(1), \dots, \ell(k)\}$, $k \in \mathbb{N}$, generated by a bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$, are uniformly bounded, then the sequence $(E_{\ell(k)})_{k=1}^{\infty}$ is a RUC basis in the space $\mathscr{L}_{0,1}$.*

$\triangleleft$ First, combining Lemma 5 and Theorem 2, we conclude that the sequence $(E_{\ell(k)})$ is a basis in $\mathscr{L}_{0,1}$. So it suffices to prove only that $\{E_{ij}\}$ is a RUC system.

Set $r_{ij}(s,t) := (-1)^{[2^i s]+[2^j t]}$, where $s, t \in I := [0,1]$. It is easily seen that, for any $n \in \mathbb{N}$,

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij} \right\|_{\mathscr{L}_{0,1}} = \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} r_{ij} \right\|_{L^\infty(I\times I)}. \tag{10}$$

From [12, Theorem 2] it follows that the system $\{r_{ij}\}_{i,j=1}^{\infty}$ possesses the RUC-property in the space $L^\infty(I \times I)$. In view of (10), the same is valid for the system $\{E_{ij}\}$ in $\mathscr{L}_{0,1}$. $\triangleright$

In particular, we get the following result, which was formulated without a proof after Corollary 3 in the paper [12].

Corollary 2. *The sequence $(E_{ij})_{i,j=1}^{\infty}$ taken in the rectangular order is a RUC basis in the space $\mathscr{L}_{0,1}$.*

4. Uniform Boundedness of $\{P_{M(k)}\}$: a Necessary Condition

In this section we show that, in contrast to rectangular projections, triangular ones in general fail to be uniformly bounded. For the reader's convenience, we present below a slight modification of the results, which are already known (see, e. g., [11]).

It is well known that the Hilbert matrix $(h_{ij})_{i,j=1}^{\infty}$ with the elements

$$h_{ij} = \begin{cases} (i-j)^{-1}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases} \tag{11}$$

defines a bounded operator $H\colon l^2 \to l^2$. The Toeplitz theorem asserts that its norm is equal to $\|h\|_{L^\infty[-\pi,\pi]} = \pi$, where

$$h(x) := \sum_{k=1}^{\infty} \frac{2\sin(kx)}{k} = \pi \operatorname{sign} x - x$$

(see, e. g., [19, Theorem 1.1] and [2, § 5.1]). Let us also note that the matrix (operator) H is a discrete analogue of the famous Hilbert transform (see, e. g., [2, Chapter 5]).

Let $n \in \mathbb{N}$. A subset $Q \subset \mathbb{N}^2$ will be called a *square mask of size n* if, by permutations of rows and columns, it can be transformed into the square $Q_n := \{1, \dots, n\}^2$.

Similarly, we will say that a subset $T \subset \mathbb{N}^2$ is a *triangular mask of size n* if there exist permutations of rows and columns in $\mathbb{N}^2$ that transform T to the form

$$T_n = \left\{(i,j) \in \mathbb{N}^2 :\ j \leqslant i \leqslant n\right\}.$$

The next result is very close to Proposition 2.1 proved by Kwapień and Pełczyński in [11].

Lemma 6. *Let $n \in \mathbb{N}$. Suppose that $H = (h_{ij})_{i,j=1}^{n}$ is the Hilbert matrix defined in* (11). *Then, the norms of the operators $P_{T_n}(H)$ and $P_{Q_n}(H)$ in the space $\mathscr{L}_{0,1}$ satisfy, respectively, the following estimates*:

$$\|P_{T_n}(H)\| > n \ln n - n, \quad \|P_{Q_n}(H)\| \leqslant \pi n.$$

$\lhd$ Obviously,

$$\|P_{T_n}(H)\| = \sum_{j<i\leqslant n} \frac{1}{i-j} = \sum_{k=1}^{n-1} \frac{n-k}{k} > n \ln n - n + 1.$$

On the other hand, as was noted above, the norm of the operator $H\colon l^2 \to l^2$ is equal to π. Hence, for any $x \in c_0$ we have

$$\|P_{Q_n}(H)\, x\|_{l^1} = \|\pi_n H \pi_n\, x\|_{l^1} \leqslant \sqrt{n} \|\pi_n H \pi_n\, x\|_{l^2} \leqslant \sqrt{n}\, \pi\, \|\pi_n\, x\|_{l^2} \leqslant \pi\, n\, \|x\|_{l^\infty}.$$

This implies that $\|P_{Q_n}(H)\| \leqslant \pi n$. $\rhd$

Since the norm of an operator $A \in \mathscr{L}_{0,1}$ is preserved under any permutations of the rows and columns of its matrix, we obtain the following result.

Theorem 4. *Let $\ell\colon \mathbb{N} \to \mathbb{N}^2$ be a bijection and $M(k) = \{\ell(1), \ldots, \ell(k)\}$, $k \in \mathbb{N}$, be the corresponding sequence of masks. If for every positive integer n there exist a square mask Q of size n and a positive integer $k = k(n)$, such that the intersection $Q \cap M(k)$ is a triangular mask T of size n, then the sequence $(E_{\ell(k)})$ fails to be a basis in $\mathscr{L}_{0,1}$.*

$\lhd$ Indeed, for each $n \in \mathbb{N}$, by Lemma 6, we can choose an operator $H' = P_Q(H')$, such that

$$\|P_{M(k)}\| \geqslant \frac{\|P_{M(k)}(H')\|}{\|H'\|} = \frac{\|P_T(H')\|}{\|P_Q(H')\|} > \frac{\ln n - 1}{\pi}.$$

Hence, the projections $P_{M(k)}$ are not uniformly bounded, and, according to Theorem 2, the sequence $(E_{\ell(k)})$ is not a basis in $\mathscr{L}_{0,1}$. $\rhd$

Theorem 4 implies an interesting corollary. It turns out that being non-basic is typical for the random numbering $\ell\colon \mathbb{N} \to \mathbb{N}^2$. Consider a primitive set

$$S = \bigcup_{k=1}^{\infty} \beta_k \times \alpha_k,$$

provided that $\min\{|\beta_k|, |\alpha_k|\} \to \infty$ as $k \to \infty$. Let us call such a primitive set *expanding*. With every expanding primitive set S we associate a random permutation $\tau_S\colon \mathbb{N}^2 \to \mathbb{N}^2$ that randomly and independently permutes pairs (i, j) inside each "rectangle" $\beta_k \times \alpha_k$. Then the following statement holds.

Corollary 3. *For every expanding primitive set S and any bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$, the sequence $(E_{\tau_S(\ell(k))})$ is not a basis in $\mathscr{L}_{0,1}$ with probability one.*

$\lhd$ For every $n \in \mathbb{N}$, let us place in S an infinite system of pairwise disjoint "squares" (square masks) of size n. The elements of each of these "squares" are traversed by the sequence $\tau_S(\ell(k))$ so that all $(n^2)!$ possible orderings of each "square" are equally probable. In addition, the orderings in different "squares" are independent. During the ordering process, a triangular mask of size n appears in each "square" with a certain positive probability p_n. Since there are infinitely many of those "squares", a triangular mask of size n will appear in one of them, sooner

or later, with probability one. And since n is arbitrary, the random bijection $k \mapsto \tau_S(\ell(k))$ satisfies the conditions of Theorem 4 almost surely. $\triangleright$

To get an extension of Theorem 4 we introduce one more definition.

If elements of a square matrix $n \times n$ by means of permutations of rows and columns can be represented in the form (11), this matrix will be called Hilbert as well.

Theorem 5. *Suppose $\ell\colon \mathbb{N} \to \mathbb{N}^2$ is a bijection and $M(k) = \{\ell(1), \dots, \ell(k)\}$, $k \in \mathbb{N}$, is the corresponding sequence of masks. If for any $C > 0$ there exist positive integers n, k, a square mask Q of size n, and a Hilbert matrix $(h_{ij})_{(i,j)\in Q}$, such that*

$$\sum_{(i,j)\in Q\cap M(k)} h_{ij} > Cn,$$

then the sequence $(E_{\ell(k)})$ fails to be a basis in $\mathscr{L}_{0,1}$.

It is proved similar to Theorem 4, by using the estimate $\|P_{M(k)}\| > C/\pi$.

Remark 1. Arguing as in the proof of Proposition 1.1 in [11], one can check that a triangular mask of size n may be represented as a union of no more than $\log_2 n + 1$ primitive sets (for an extension of this result see [20]).

5. A Special Example of the Sequence of Masks

In view of the results obtained in the previous sections, the following problem arises naturally.

Problem. What happens if a bijection $\ell\colon \mathbb{N} \to \mathbb{N}^2$ does not satisfy the conditions of either Theorems 3 and 4? In this case, can the sequence $(E_{\ell(k)})$ be a basis in $\mathscr{L}_{0,1}$?

Leaving the question open, we assert that bijections ℓ of the above mentioned type do exist. More precisely, there exist bijections $\ell\colon \mathbb{N} \to \mathbb{N}^2$ such that the ranks of the corresponding masks $M(k) = \{\ell(1), \dots, \ell(k)\}$ take arbitrarily large values, and at the same time, none of the triangle masks of size 5 can be obtained as the intersection of a square mask Q and some $M(k)$. To prove this, we construct below examples of masks with arbitrarily large ranks that however contain no square mask of size 2 and, consequently, no triangular mask of size 3. Using them as patterns, one can construct the desired sequence $(M(k))_{k=1}^{\infty}$. We leave details of this geometric construction to the reader.

Let p be a prime number. Denote by $\{n\}_p$ the residue of division of the integer n by p and define the following p^2 sets:

$$\alpha_{ij} = \big\{1 + \{i + kj\}_p + kp :\ k = 0, \dots, p-1\big\}, \quad i, j = 0, \dots, p-1.$$

Consider the intersection of any two of them, say, α_{ij} and $\alpha_{i'j'}$. Assuming that

$$1 + \{i + kj\}_p + kp = 1 + \{i' + k'j'\}_p + k'p$$

for some $k, k' \in \{0, \dots, p-1\}$, we get $k' = k$, and hence the expression $i - i' + k(j - j')$ is divisible by p. If $j \neq j'$, then the latter holds for a unique k (because the numbers $i - i' + k(j - j')$, $k = 0, 1, \dots, p-1$, have different residues modulo p). Consequently, the sets α_{ij} and $\alpha_{i'j'}$ have exactly one common element if $j \neq j'$. And if $j = j'$, these sets do not intersect at all.

Now we define the mask M_p by

$$M_p = \bigcup_{j=0}^{p-1} \bigcup_{i=0}^{p-1} \beta_{ij} \times \alpha_{ij}, \quad \text{where} \quad \beta_{ij} = \{1 + i + jp\}.$$

By construction, the set M_p possesses the following properties:
(a) it contains no square mask of size 2 and hence no triangular mask of size 3;
(b) it is contained in the square mask $Q_{p^2} = \{1, \dots, p^2\}^2$;
(c) its cardinality is p^3;
(d) it can be partitioned into p primitive sets N_j of the form

$$N_j := \bigcup_{i=0}^{p-1} \beta_{ij} \times \alpha_{ij}, \quad j = 0, 1, \dots, p-1.$$

On the other hand, we prove below that the rank of M_p tends to infinity together with p.

First, let us verify that M_p cannot be represented as a union of $(p-1)/2$ primitive sets. Indeed, let $N = \bigcup_i \gamma_i \times \delta_i$ be an arbitrary primitive set that is contained in M_p. By property (a) of M_p, none of its parts $\gamma_i \times \delta_i$ contains a square mask of size 2. Hence, either $|\gamma_i| = 1$ or $|\delta_i| = 1$. By the definition of primitive sets, all γ_i (resp. δ_i) are pairwise disjoint. Consequently, we have

$$|N| = \sum_i |\gamma_i \times \delta_i| < \sum_i \bigl(|\gamma_i| + |\delta_i|\bigr) = \sum_i |\gamma_i| + \sum_i |\delta_i| \leqslant 2p^2.$$

Combining this estimate together with property (c), we deduce the desired result.

In view of the latter claim, it seems plausible that the rank of M_p is at least $p/2$. However, we are only able to prove a weaker result that this rank is asymptotically not less than $\ln p \,/\, \ln\ln p$. We derive this estimate by using the following proposition, which will be proved at the very end of this section.

Proposition 1. *Let Q be a square mask of size n. Then, any subset $M \subset Q$ of rank r can be partitioned into at most $(\log_2 n + 4)^r$ primitive subsets.*

Taking into account the above observation and applying Proposition 1 to the set M_p and the square mask Q_{p^2} (see property (b)), we obtain the inequality

$$\bigl(\log_2 p^2 + 4\bigr)^r > p/2,$$

which implies the desired lower estimate for the rank r of M_p:

$$r > \frac{\ln(p/2)}{\ln(\log_2 p^2 + 4)} > \frac{\ln p - \ln 2}{\ln(4 \ln p + 4)} > \frac{\ln p - \ln 2}{\ln\ln p + \ln 8}, \quad p \geqslant 3.$$

Now, we proceed with proving Proposition 1. First, we prove some auxiliary results. As above, sets α_i, β_i in the representation $M = \bigcup_i \beta_i \times \alpha_i$ of a primitive set M are allowed to be empty.

Lemma 7. *The intersection of two primitive sets is primitive as well.*

$\lhd$ Let us consider two primitive sets

$$M = \bigcup_i \beta_i \times \alpha_i \quad \text{and} \quad M' = \bigcup_j \beta'_j \times \alpha'_j.$$

Then, obviously,

$$M \cap M' = \bigcup_{ij} (\beta_i \cap \beta'_j) \times (\alpha_i \cap \alpha'_j).$$

Since the sets $\beta_i \cap \beta'_j$ (resp. $\alpha_i \cap \alpha'_j$) are pairwise disjoint (or empty), the set $M \cap M'$ is primitive, and so the proof is completed. $\rhd$

Lemma 8. *Suppose that* $M = \bigcup_{i=1}^{2^n} \beta_i \times \alpha_i$ *is a primitive set, and*

$$\beta := \bigsqcup_{i=1}^{2^n} \beta_i, \quad \alpha := \bigsqcup_{i=1}^{2^n} \alpha_i.$$

Then the set $(\beta \times \alpha) \setminus M$ *can be partitioned into* n *primitive sets.*

$\lhd$ We proceed by induction. First, for $n = 1$, we have $\alpha = \alpha_1 \sqcup \alpha_2$ and $\beta = \beta_1 \sqcup \beta_2$. In this case, the set $(\beta \times \alpha) \setminus M$ is primitive, because

$$(\beta \times \alpha) \setminus M = \beta_1 \times \alpha_2 \sqcup \beta_2 \times \alpha_1.$$

Assuming now that the assertion of the lemma holds for some n, let us prove it for $n+1$. Given $M = \bigcup_{i=1}^{2^{n+1}} \beta_i \times \alpha_i$, we set

$$\beta' = \bigsqcup_{i=1}^{2^n} \beta_i, \quad \beta'' = \bigsqcup_{i=1}^{2^n} \beta_{i+2^n}, \quad \alpha' = \bigsqcup_{i=1}^{2^n} \alpha_i, \quad \alpha'' = \bigsqcup_{i=1}^{2^n} \alpha_{i+2^n},$$

and also set $M' = (\beta' \times \alpha') \cap M$ and $M'' = (\beta'' \times \alpha'') \cap M$. Then, we have

$$(\beta \times \alpha) \setminus M = \big((\beta' \times \alpha') \setminus M'\big) \sqcup \big((\beta'' \times \alpha'') \setminus M''\big) \sqcup \big((\beta' \times \alpha'') \sqcup (\beta'' \times \alpha')\big). \tag{12}$$

Note that either of the first and second sets from the right-hand side of equality (12) can be partitioned, by the induction hypothesis, into n primitive sets. Let us combine elements of these partitions in n pairs, taking in each pair one element from the first partition and the other from the second partition. The unions of members of each pair are obviously primitive. Adding to them the third set from (12), which is primitive itself, we obtain a partition of the difference $(\beta \times \alpha) \setminus M$ into $n+1$ disjoint primitive sets. $\rhd$

Lemma 9. *For any square mask* Q *of size* n *and primitive set* M*, the difference* $Q \setminus M$ *can be partitioned into* $[\log_2 n] + 3$ *primitive sets.*

$\lhd$ By Lemma 7, the intersection $Q \cap M$ is primitive, i. e.,

$$Q \cap M = \bigcup_{i=1}^{m} \beta_i \times \alpha_i, \tag{13}$$

where α_i (resp. β_i) are pairwise disjoint. Note that $m \leqslant n$. Taking now

$$\beta = \bigsqcup_{i=1}^{m} \beta_i \quad \text{and} \quad \alpha = \bigsqcup_{i=1}^{m} \alpha_i,$$

we see that $Q \cap M \subset \beta \times \alpha \subset Q$.

Supplementing (if necessary) representation (13) with empty sets, we can assume that m is equal to 2^k, where the integer k is such that $2^{k-1} < n \leqslant 2^k$. Then, by Lemma 8, the difference $(\beta \times \alpha) \setminus M$ can be partitioned into k primitive sets. In turn, the difference $Q \setminus (\beta \times \alpha)$ can be partitioned into two primitive sets. As a result, we infer that the set

$$Q \setminus M = \big(Q \setminus (\beta \times \alpha)\big) \sqcup \big((\beta \times \alpha) \setminus M\big)$$

is a union of $k+2$ disjoint primitive sets. Since $k + 2 < \log_2 n + 3$, everything is done. $\rhd$

Finally, we prove Proposition 1. Fix a square mask Q of size n. Consider the trace of any rank-one algebra on Q. Clearly, it is an algebra itself, which by Lemmas 7 and 9 can be embedded in some algebra of subsets of Q generated by a partition of Q into $[\log_2 n] + 4$ primitive atoms. In turn, this implies that the trace on Q of any algebra of rank r can be embedded into an algebra of subsets of Q generated by a partition of Q into $([\log_2 n] + 4)^r$ primitive atoms (some of which might be empty). This argument implies Proposition 1.

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Received February 6, 2026

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Владикавказский математический журнал
2026, Том 28, Выпуск 2, С. 5–19

УСЛОВНЫЕ БАЗИСЫ В ПРОСТРАНСТВЕ МАТРИЦ С КАТ-НОРМОЙ

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Аннотация. В работе рассматривается пространство $\mathscr{L}_{0,1}$ бесконечных матриц вида $(a_{ij})_{i,j=1}^{\infty}$, где $a_{ij} \in \mathbb{R}$, снабженное кат-нормой. Кат-норма матрицы естественным образом появляется в разных разделах дискретной математики, но используется и в функциональном анализе. Она эквивалентна операторной норме матрицы, рассматриваемой как отображение из пространства c_0 сходящихся к нулю

последовательностей (или из l^∞) в l^1. В работе доказано, что множество матричных единиц $\{E_{ij}\}_{i,j=1}^{\infty}$ образует систему случайной безусловной сходимости в пространстве $\mathscr{L}_{0,1}$, но при этом не является системой безусловной сходимости. Эта система полна в $\mathscr{L}_{0,1}$, и при некотором упорядочивании может претендовать на роль условного базиса. В работе получены как необходимое, так и достаточное условия на биекцию $\ell\colon \mathbb{N} \to \mathbb{N}^2$, при выполнении которых последовательность $(E_{\ell(k)})$ образует базис в $\mathscr{L}_{0,1}$. В частности, любое последовательное заполнение угловых квадратов приводит к базису, в то время как треугольное заполнение — нет. Показано, что случайные перестановки последовательности $(E_{\ell(k)})$ почти наверное не образуют базис. Поставлена проблема полной характеризации тех биекций $\ell\colon \mathbb{N} \to \mathbb{N}^2$, которые приводят к базису $(E_{\ell(k)})$, и конструктивно показано, что предложенные необходимое и достаточное условия не дают полного ответа на эту проблему.

Ключевые слова: условный базис, система случайной безусловной сходимости, пространство бесконечных матриц, операторная норма, кат-норма.

AMS Subject Classification: 46B15, 46B28, 46B45.

Образец цитирования: *Astashkin S. V., Bakhtin V. I., Lykov K. V.* Conditional Bases in the Matrix Space with the Cut-Norm // Владикавк. мат. журн.—2026.—Т. 28, № 2.—С. 5–19 (in English). DOI: 10.46698/b6298-9094-1158-n.