Vladikavkaz Mathematical Journal
2025, Volume 27, Issue 4, P. 148–156

УДК 517.98
DOI 10.46698/x9860-3651-6483-z

WHAT REMAINS THE SAME IN ORDER CONVERGENCE TYPES

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Abstract. In this paper, we examine what remains the same between order convergence and unbounded order convergence, as well as between unbounded order continuity and strongly unbounded order continuity. In [1], Gao et al. proved that a sublattice of a Riesz space is order closed if and only if it is unbounded order closed. It is shown that σ-ideals and unbounded σ-ideals are the same. Additionally, it is established that injective band operators are unbounded order continuous, while bijective order bounded disjoint preserving operators are order continuous. Let G be an order dense majorizing Riesz subspace of a Riesz space E, and let F be a Dedekind complete Riesz space. In reference [2], the question is posed: If $T : G \to F$ is a positive strongly unbounded order continuous operator, does T have a unique positive strongly unbounded order continuous extension to all of E? We prove that this problem has a positive answer whenever G is *suo*-convergence reducing of E, namely, if $x_\alpha \overset{suo}{\to} 0$ in E then $x_\alpha \overset{uo}{\to} 0$ in G for any net (x_α) in G.

Keywords: unbounded order convergent, unbounded order closed ideal, unbounded order continuous operator, strongly unbounded order continuous operator.

AMS Subject Classification: 46A40, 47B65.

For citation: *Uyar, A.* What Remains the Same in Order Convergence Types, *Vladikavkaz Math. J.*, 2025, vol. 27, no. 4, pp. 148–156. DOI: 10.46698/x9860-3651-6483-z.

1. Introduction and Preliminaries

In classical literature such as [3, 4], the definition of order convergence are slightly different. In [5], various types of order convergence are investigated. We will consider two of them. A *net* of a set X is a mapping $u : \Gamma \to X$ from a directed set Γ to X. As usual, $u(\alpha)$ is denoted by x_α and the net $u : \Gamma \to X$ by (x_α), omitting the directed set Γ if its presence is not necessary.

Let E be the Riesz space. A net $(x_\alpha)_{\alpha\in\Gamma}$ in E is:

(a) *order convergent* (*o-convergent*) to $x \in E$ if there is another net $(z_\beta)_{\beta\in\Lambda}$ in E with $z_\beta \downarrow 0$ and for every β, there exists α_0 such that $|x_\alpha - x| \leqslant z_\beta$ for all $\alpha \geqslant \alpha_0$. In this case, we write $x_\alpha \overset{o}{\to} x$.

(b) *strongly order convergent* (*so-convergent*) to $x \in E$ if there is a net $(y_\alpha)_{\alpha\in\Gamma}$ in E such that $(y_\alpha) \downarrow 0$ and $|x_\alpha - x| \leqslant y_\alpha$ for some α_0 and for all $\alpha \geqslant \alpha_0$. In this case, we write $x_\alpha \overset{so}{\to} x$.

Obviously, $x_\alpha \overset{so}{\to} x$ implies $x_\alpha \overset{o}{\to} x$. In general, o-convergence of a sequence does not imply *so*-convergence (see [5, Example 1.4]). If E is the Dedekind complete Riesz space, then strongly order convergence is equivalent to order convergence. A net (x_α) is *unbounded order convergent* (*uo-convergent*) to $x \in E$, written as $x_\alpha \overset{uo}{\to} x$, if $|x_\alpha - x| \wedge u \overset{o}{\to} 0$ for all $u \in E_+$.

Similarly, strongly unbounded order convergence is defined. In this case, we write $x_\alpha \overset{suo}{\to} x$. It is clear that $x_\alpha \overset{suo}{\to} x$ implies $x_\alpha \overset{uo}{\to} x$.

Recall that a net (x_α) in a Riesz space E is said to be *order bounded* if there exists $x \in E_+$ such that $|x_\alpha| \leqslant x$ for all α. For any net (x_α) in E,

(i) $x_\alpha \overset{uo}{\to} 0$ iff $|x_\alpha| \overset{uo}{\to} 0$;

(ii) $x_\alpha \overset{o}{\to} x$ iff $x_\alpha \overset{uo}{\to} x$ and a tail of (x_α) is order bounded;

(iii) $x_\alpha \overset{so}{\to} 0$ iff $x_\alpha \overset{suo}{\to} x$ and a tail of (x_α) is order bounded.

Note that the *uo*-limit is unique whenever it exists. Let $\mathbb{R}^X$ be the Riesz space of all real-valued functions on a non-empty set X, equipped with the pointwise order. It is not difficult to see that a net x_α in $\mathbb{R}^X$ *uo*-converges to $x \in \mathbb{R}^X$ iff it converges pointwise to x. For general results on *uo*-convergence we refer the reader to [1, 6, 7].

An order closed ideal is referred to as *band*. Let A be a nonempty subset of the Riesz space E. Then the band generated by A is the smallest band that contains A. The band generated by a set A and an element x is denoted by B_A and B_x , respectively. We write A^{dd} for $(A^d)^d$ where A^d is *disjoint complement* of A, namely, $A^d = \{x \in E : \ x \perp y \text{ for all } y \in A\}$. The band generated by a nonempty subset A of an Archimedean Riesz space is A^{dd}. The ideal A in E is called *σ-ideal* whenever it follows from $(x_n) \subset A$ and $\sup x_n = x$ in E that $x \in E$.

All Riesz spaces are assumed to be Archimedean. For terminology and results on Riesz spaces not explained in this paper we refer to [3, 4, 8, 9].

In [10], unbounded order continuous operators are introduced. An operator $T : E \to F$ between two Riesz spaces is said to be *unbounded order continuous* (*uo-continuous*), if $x_\alpha \overset{uo}{\to} 0$ in E implies $Tx_\alpha \overset{uo}{\to} 0$ in F. The definition of the *strongly unbounded order continuous* (*suo-continuous*) operator is similar. In particular, lattice operations are unbounded order continuous (see [7, Lemma 3.1]).

It is known that the sublattice of a Riesz space is order closed if and only if it is unbounded order closed (see [1, Proposition 3.15]). In section 2 of this paper, a simpler proof of this proposition is given, taking the ideal instead of the sublattice. Moreover, it is obtained that the class of σ-ideals is the same as the class of unbounded σ-ideals. In section 3, it is demonstrated that an injective principal band operator is *uo*-continuous, and a bijective order bounded disjoint preserving operator is order continuous. In section 4, we aim to provide a partial solution to the open problem given in [2, Problem 7].

2. Unbounded Order Closed Ideals

In [11], Bilokopytov presents a characterization of the *uo*-convergence as follow:

Proposition 1 [11, Theorem 6.4]**.** *Let $(x_\alpha) \subset E$ and $x \in E$. Then $x_\alpha \overset{uo}{\to} x$ iff*

(i) $\inf_{\alpha' \geqslant \alpha}(x \vee x_{\alpha'}) = x = \sup_{\alpha' \geqslant \alpha}(x \wedge x_{\alpha'})$ *for every α,*

and

(ii) *if $y \in E$ and $\inf_{\alpha' \geqslant \alpha}(y \vee x_{\alpha'}) = y = \sup_{\alpha' \geqslant \alpha}(y \wedge x_{\alpha'})$ for every α, then $y = x$ hold.*

Corollary 1 [12, Lemma 3.1]**.** *Let E be a Riesz space and $(x_\alpha) \subset E$ such that $x_\alpha \overset{uo}{\to} 0$. Then* $\inf |x_\alpha| = 0$.

$\lhd$ It follows from Lemma 3.1 in [7] $|x_\alpha| \overset{uo}{\to} 0$. By Proposition 1, $\inf_{\alpha' \geqslant \alpha} |x_{\alpha'}| = 0$ for every α. It is easily seen that $\inf |x_\alpha| = 0$. $\rhd$

Let E be a Riesz space and $(x_\alpha) \subset E$. It is said that (x_α) is *eventually null net* if there is α_0 such that $x_\alpha = 0$, for every $\alpha \geqslant \alpha_0$. Every element of c_{00} is eventually null net. From Theorem 3.2 in [1], if A is either an ideal or order dense sublattice of Riesz space E, then for a net (x_α) in A, $x_\alpha \overset{uo}{\to} 0$ in A iff $x_\alpha \overset{uo}{\to} 0$ in E. Thus, for a net (x_α) in E, $x_\alpha \overset{uo}{\to} 0$ in E

iff $x_\alpha \overset{uo}{\to} 0$ in a Dedekind complete of E [1]. Therefore, without loss of generality, we may assume that E is Dedekind complete in some *uo*-convergence studies.

Proposition 2. *Let E be a Riesz space and $(x_\alpha)_{\alpha\in\Gamma} \subset E$.*

(i) *Recall that, an element $\alpha_0 \in \Gamma$ is said to be maximal whenever $\alpha_0 \leqslant \alpha$ for any $\alpha \in \Gamma$, then $\alpha_0 = \alpha$. If the index set Γ has a maximal element and $x_\alpha \overset{uo}{\to} 0$, then (x_α) eventually null net.*

(ii) *If E is a Riesz space and $(x_\alpha)_{\alpha\in\Gamma}$ is a disjoint net in E such that the index set Γ doesn't have a maximal element then $x_\alpha \overset{uo}{\to} 0$.*

$\lhd$ (i) Suppose that the index set Γ has a maximal element and let this element be α_0. Let $x_\alpha \overset{uo}{\to} 0$ and $u = |x_{\alpha_0}|$. Then $|x_\alpha| \wedge u \overset{o}{\to} 0$. There is another net (z_β) in E such that $z_\beta \downarrow 0$ and that for any β, there exists α_1 such that $|x_\alpha| \wedge u \leqslant z_\beta$ for all $\alpha \geqslant \alpha_1$. Let's take an arbitray β and consider α_1 corresponding to this β. Let's choose the index α_2, which yields $\alpha_0 \leqslant \alpha_2$ and $\alpha_1 \leqslant \alpha_2$. For every $\alpha \geqslant \alpha_2$, we have $x_\alpha = x_{\alpha_0}$ and so $|x_{\alpha_0}| = |x_\alpha| \wedge u \leqslant z_\beta$. Thus, $|x_{\alpha_0}| \leqslant z_\beta$ for every β. Then we have $x_\alpha = 0$ for every $\alpha \geqslant \alpha_0$.

(ii) Let's assume that E is a Dedekind complete. Fix $u \in E_+$. We claim that $0 \leqslant \sup_{\alpha'\geqslant\alpha}(|x_{\alpha'}| \wedge u) \downarrow_\alpha 0$. Suppose that $\sup_{\alpha'\geqslant\alpha}(|x_{\alpha'}| \wedge u) \geqslant y \geqslant 0$ for all α. Since Γ doesn't have a maximal element, Γ has an element α_0 such that $\alpha < \alpha_0$ for each α. Then,

$$0 \leqslant y \wedge |x_\alpha| \leqslant \left(\sup_{\alpha'\geqslant\alpha_0}(|x_{\alpha'}| \wedge u)\right) \wedge |x_\alpha| = \sup_{\alpha'\geqslant\alpha_0}(|x_{\alpha'}| \wedge |x_\alpha| \wedge u) = 0.$$

Thus, we have $y \wedge |x_\alpha| = 0$ for all α. Let choose $\alpha_1 \in \Gamma$. It follows that

$$y = y \wedge \sup_{\alpha'\geqslant\alpha_1}(|x_{\alpha'}| \wedge u) = \sup_{\alpha'\geqslant\alpha_1}(y \wedge |x_{\alpha'}| \wedge u) = 0.$$

Now it is immediately that $|x_\alpha| \wedge u \overset{o}{\to} 0$. $\rhd$

The above proposition is analogue to Corollary 3.6 in [1]. The condition in the proposition that the index set does not have a maximal element cannot be removed.

Example 1. Let $X = [0, 1]$ and $f_\alpha \in \mathbb{R}^X$ be denoted the characteristic function of $\{\alpha\}$ for each $\alpha \in X$. It is easy to see that $(f_\alpha)_{\alpha\in X}$ is a disjoint net but not *uo*-converge to 0.

Let us present the routine definition of unbounded order closed sets.

Definition 1. Let E be a Riesz space.

(i) A subset A of E is said to be *unbounded order closed* (*uo-closed*) whenever $(x_\alpha) \subset A$ and $x_\alpha \overset{uo}{\to} x$ imply $x \in A$,

(ii) An ideal A in E is said to be *unbounded σ-ideal* (*uσ-ideal*) in E whenever $(x_n) \subset A$ and $x_n \overset{uo}{\to} x$ imply $x \in A$,

(iii) An ideal A in E is said to be *unbounded order band* (*uo-band*) in E whenever A is *uo*-closed.

(iv) Let A be a nonempty subset of E. Then the *uo-band generated* by A is the smallest (with respect to inclusion) *uo*-band that includes A. The *uo*-band generated by a vector $x \in E$ will be denoted U_x.

Clearly, the *uo*-band generated by A coincides with the band generated by the ideal generated by A. The unbounded order continuity of the lattice operations guarantees disjoint complement of any nonempty set is a *uo*-band. Thus, we get the following simple result.

Corollary 2. *Let A be an ideal in E and $(x_\alpha) \subset A_+$, $x \in E_+$, such that $x_\alpha \overset{uo}{\to} x$ in E. Then $x \in A^{dd}$.*

$\lhd$ It follows easily that A^{dd} is the band generated by A. $\rhd$

It is clear that $x_\alpha \xrightarrow{o} 0$ in E implies $x_\alpha \xrightarrow{uo} 0$, the converse is not true. Let $E = c_0$, and let (e_n) be the standard basis of E. Then (e_n) *uo*-converges to 0, but it is not order convergent to zero. However, the sublattice of a Riesz space is order closed if and only if it is unbounded order closed (see [1, Proposition 3.15]). We obtain a simpler proof of this result when we work with ideals instead of sublattices.

Proposition 3. *Let E be a Riesz space and A be an ideal in E. Then A is uo-closed if and only if A is o-closed. In other words, A is a band iff A is a uo-band.*

$\lhd$ The "only if" part is straightforward since order convergent nets are *uo*-convergent.

For the "if" part, suppose A is a *o*-closed. Then A is a band. If U_A is a *uo*-band generated by A, then $U_A \subset A^{dd}$ because A^{dd} is a *uo*-band. Since $A = A^{dd}$ (see [8, Theorem 1.28]), $A = U_A$ and so A is *uo*-closed. $\rhd$

In the case of sequentially closeness, we will use the characterization of *uo*-convergence that we gave at the beginning of this section.

Proposition 4. *A is a $u\sigma$-ideal in a Riesz space E if and only if A is a σ-ideal in E.*

$\lhd$ The "only if" part straightforward since order convergent sequences are *uo*-convergent.

For the "if" part, suppose A is a σ-ideal, and let $(x_n) \subset A$, $x \in E$ and $x_n \xrightarrow{uo} x$ in E. Then $|x_n| \xrightarrow{uo} |x|$ in E by [7, Lemma 3.1]. Thus, without loss of generality, we assume $(x_n) \subset A_+$ and $x \in E_+$. By Theorem 6.4 of [11], $\sup(x \wedge x_n) = x$. Since $x \wedge x_n \in A$ for every n, we have $x \in A$. $\rhd$

A sublattice A of a Riesz space E is said to be *dense with respect to order convergence* (*uo-convergence*) if every vector in E is order limit (unbounded order limit) of a net in A. Then, A is dense with respect to order convergence iff A is dense with respect to unbounded order convergence. Indeed, it is clear that if A is dense with respect to order convergence then A is dense with respect to unbounded order convergence. On the other hand, we have $x_\alpha \xrightarrow{uo} x$ then $(|x_\alpha - x| + x) \wedge (|x| + x) \xrightarrow{o} x$. Thus, if A is dense with respect to unbounded order convergence then it is dense with respect to order convergence.

3. Unbounded Order Continuous Operators

In [10, 13], *uo*-continuous operators were introduced and investigated. Then, some properties of *uo*-continuous operators were studied in [2, 12]. Recall that an operator $T : E \to F$ between two Riesz spaces is said to be *unbounded order continuous* (*uo-continuous*), if $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{uo} 0$ in F. The definition of the strongly unbounded order continuous (*suo*-continuous) operator is similar.

It is well known that every order continuous operator is order bounded [5, Theorem 2.1]. It is easy to see that every *uo*-continous functional is order continuous and so order bounded. It is natural to ask whether *uo*-continuous operators are order bounded:

Question 1. Is every *uo*-continuous (*suo*-continuous) operator order bounded?

Let $T : E \to F$ be an order bounded operator between two Riesz space with F Dedekind complete. If T is a *uo*-continuous (*suo*-continuous) operator then $|T|$ is a *uo*-continuous (*suo*-continuous) (see [2, Theorem 2]). This theorem allows us to present an alternative and shorter proof of Proposition 3.2 of [12].

Proposition 5. *Let $T : E \to F$ be an order bounded operator between two Riesz spaces with F Dedekind complete.*

(i) *If T is a uo-continuous operator, then T^+, T^-, T and $|T|$ is an o-continuous.*

(ii) *If T is an suo-continuous operator, then T^+, T^-, T and $|T|$ is an o-continuous.*

$\lhd$ (i) It is enough to show that T^+ is an *o*-continuous operator. Let $x_\alpha \xrightarrow{o} 0$ in E. Then, $x_\alpha \xrightarrow{uo} 0$ and a tail of (x_α) is order bounded. Since T^+ is a positive *uo*-continuous operator, $T^+x_\alpha \xrightarrow{uo} 0$ and a tail of (T^+x_α) is order bounded. Therefore $T^+x_\alpha \xrightarrow{o} 0$.

(ii) The proof is also valid, with minor modification, for the *o*-continuity of T^+. However, let us give a shorter proof. Let $x_\alpha \downarrow 0$ in E. Then $T^+x_\alpha \xrightarrow{suo} 0$. From Corollary 1, $T^+x_\alpha \downarrow 0$. $\rhd$

Proposition 6. Let $T : E \to F$ be a lattice homomorphism between two Riesz spaces. The following are equivalent:

(1) T is *uo*-continuous;

(2) T is *o*-continuous;

(3) T is *suo*-continuous.

$\lhd$ By [12, Theorem 3.3], (1) $\Leftrightarrow$ (2) and by Proposition 5, (3) $\Rightarrow$ (2) hold. It is enough to show that (2) $\Rightarrow$ (3). Let T be a *suo*-continuous and $x_\alpha \downarrow 0$. Then we have $T^+x_\alpha \xrightarrow{suo} 0$ and so $T^+x_\alpha \xrightarrow{uo} 0$. By Corollary 1, $\inf Tx_\alpha = 0$. Therefore, we obtain that $Tx_\alpha \downarrow 0$. $\rhd$

As a consequence of the preceding proposition, we obtain the following result.

Corollary 3. *Let$T : E \to F$ be an order bounded disjoint preserving operator between two Riesz spaces. The following are equivalent:*

(1) *T is uo-continuous;*

(2) *T is o-continuous;*

(3) *T is suo-continuous.*

$\lhd$ If $T : E \to F$ is an order bounded disjoint preserving operator, then its modulus exists and $|T| : E \to F$ is a lattice homomorphism [4, Theorem 2.40]. Using Proposition 6, the proof is easily obtained. $\rhd$

The author introduced the definition of the band operator in [14]. Some properties were proved for band operators and inverse band operators, including the relations between band operators and disjointness preserving operators in [15] and [16]. Let $T : E \to F$ be an operator between two Riesz space. T is called a *band operator* if $T(B)$ is a band in F for each band B in E. T is called *principal band operator* if $B_{Tx} \subset T(B_x)$ for each $x \in E$. It is easy to see that if T is a band operator then T is a principal band operator (see [16, Corollary 5]). Let, now, we can give the next result.

Corollary 4. *Let $T : E \to F$ be an order bounded injective principal band operator between two Riesz spaces. Then T is o-continuous, uo-continuous and suo-continuous operator.*

$\lhd$ Since T is an order bounded injective principal band operator, T is order continuous by [16, Theorem 16]. From [15, Theorem 3.1], T is a disjoint preserving operator. Therefore, T is both *uo*-continuous and *suo*-continuous by Corollary 3. $\rhd$

Corollary 5. *Let $T : E \to F$ be an order bounded bijective disjoint preserving operator between two Riesz spaces. Then T and T^{-1} are o-continuous, uo-continuous and suo-continuous operators.*

$\lhd$ It is well known that T^{-1} is order bounded disjoint preserving operators (see [17, Theorem 1]). Then T and T^{-1} are order bounded band operators by Corollary 9 in [16]. As we mentioned before, both operators are order bounded bijective principal band operators. From Corollary 4, both operators are *o*-continuous, *uo*-continuous and *suo*-continuous. $\rhd$

4. On a Problem About Strongly Unbounded Order Continuous Operators

For a Riesz space E, we write E^δ for its Dedekind completion. Recall from [3, Theorem 1.41] that E^δ is the unique (up to a lattice isomorphism) Dedekind complete Riesz space that contains E as a majorizing and order dense sublattice.

Let E and F be Riesz spaces. We denote the set of all order bounded operators from E into F by $L_b(E,F)$. The space of all *uo*-continuous order bounded operators (resp. *suo*-continuous order bounded operators) is denoted by $L_{uo}(E,F)$ (resp. $L_{suo}(E,F)$. If E and F are Dedekind complete Riesz spaces, then $L_{uo}(E,F) = L_{suo}(E,F)$. If F is a Dedekind complete Riesz space, then $L_{uo}(E,F)$ is an ideal in $L_{suo}(E,F)$ (see [2]).

The space $L_{uo}(E,\mathbb{R})$ (resp. $L_{suo}(E,\mathbb{R})$) is denoted by $E_{uo}^{\sim}$ (resp. $E_{suo}^{\sim}$) and it is called *unbounded order continuous* (resp. *strongly unbounded order continuous*) dual of E. Let

$$\mathscr{L} = \left\{ \sum_{i=1}^{n} a_i f_i : \ n \in \mathbb{N},\, a_i \in \mathbb{R},\, f_i \text{ coordinate or functional on } E \right\}.$$

Then we have $\mathscr{L} = E_{uo}^{\sim} \subseteq E_{suo}^{\sim}$ by Proposition 2.5 in [12]. On the other hand, it is clear that *uo*-convergence and *suo*-convergence of a sequence to an element are the same. Thus, we obtain that $E_{suo}^{\sim} \subset \mathscr{L}$ by repeating the proof of Proposition 2.2 in the same way in [6]. Thus, the following proposition is easily obtained.

Proposition 7. *Let E be a Riesz space. Then $E_{uo}^{\sim} = E_{suo}^{\sim}$.*

In this section, our main aim is to give a partial solution to the following open problem.

Problem 1 [2, Problem 7]. Let G be an order dense majorizing sublattice of a Riesz space E, and let F be a Dedekind complete Riesz space. If $T : G \to F$ is a positive *suo*-continuous operator, does T have a unique positive *suo*-continuous extension to all of E ?

From Proposition 1 (iv) in [2], if this problem is solved positively, then $L_{uo}(E,F) = L_{suo}(E,F)$ for every Dedekind complete Riesz space F.

Definition 2. A sublattice G of a Riesz space E is said to be *o-convergence reducing of* E if $x_\alpha \overset{o}{\to} 0$ in E then $x_\alpha \overset{o}{\to} 0$ in G for any net (x_α) in G. Similarly, *uo-convergence reducing of* E and *suo-convergence reducing of* E are defined.

Let G be an order dense majorizing sublattice of a Riesz space E. Then G is an *o*-convergence reducing of E and it is also a *uo*-convergence reducing of E [1]. However, we have illustrated below that G may not be generally *suo*-convergence reducing of E:

Example 2. Let K be the one-point compactification of an uncountable discrete space, and consider the Riesz space $E = C(K)$ with point-wise order. If f_n denotes the characteristic functions of a sequence of distinct singletons in K, the sequence f_n is *o*-convergent to zero but not *so*-convergent to zero (see [5, Example 1.4]). It is clear that $f_n \overset{uo}{\to} 0$ in E implies $f_n \overset{uo}{\to} 0$ in E^δ. Since E^δ Dedekind complete, we have $f_n \overset{suo}{\to} 0$ in E. Then E doesn't *suo*-convergence reducing of E.

By Proposition 2, the following is easily obtained.

Lemma 1. (i) *An operator $T : E \to F$ between two Riesz spaces is a uo-continuous if and only if the index set Γ of (x_α) doesn't have a maximal element and $x_\alpha \overset{uo}{\to} 0$, then $Tx_\alpha \overset{uo}{\to} 0$.*

(ii) *An operator $T : E \to F$ between two Riesz spaces is a suo-continuous if and only if the index set Γ of (x_α) doesn't have a maximal element and $x_\alpha \overset{suo}{\to} 0$, then $Tx_\alpha \overset{suo}{\to} 0$.*

In the following proposition, we give the condition under which the open problem given in [2, Problem 7] has a positive answer.

Proposition 8. *Let G be an order dense majorizing sublattice of a Riesz space E, and let F be a Dedekind complete Riesz space. If G is a suo-convergence reducing of E and $T : G \to F$ is a positive suo-continuous operator, then T has a unique positive suo-continuous extension to all of E.*

$\lhd$ Let $T : G \to F$ be a positive *suo*-continuous operator. Then, $T : G \to F$ is an o-continuous operator. From Theorem 1.65 in [4], the formula

$$S(x) = \sup\{Ty : \ y \in G \text{ and } 0 \leqslant y \leqslant x\}, \quad x \in E_+$$

defines a unique positive order continuous extension $\hat{T}$ of T to all of E with $\hat{T}|E_+ = S$. It is enough to show that $\hat{T}$ is *suo*-continuous. Let $(x_\alpha)_{\alpha\in\Gamma} \subset E_+$ and $x_\alpha \overset{suo}{\to} 0$ in E. From Lemma 1, we assume that the index set Γ does not have a maximal element. Let $\Lambda_\alpha = \{(\alpha, y) : y \in G\cap[0, x_\alpha]\}$ for each α and $\Lambda = \bigcup \Lambda_\alpha$. Clearly, Λ is directed upward set with respect to the lexicographical order by defining that $(\alpha_1, y_1) \leqslant (\alpha_2, y_2)$ if and only if $\alpha_1 < \alpha_2$ or ($\alpha_1 = \alpha_2$ and $y_1 \leqslant y_2$). Let us set $x_{(\alpha,y)} = y$. Then we have $(x_{(\alpha,y)}) \subset G$. Let's take $u \in E_+$. There is a net (y_α) in E such that $y_\alpha \downarrow 0$ and $x_\alpha \wedge u \leqslant y_\alpha$ for some α_0 and for all $\alpha \geqslant \alpha_0$. Then we have

$$x_{(\alpha,y)} \wedge u = y \wedge u \leqslant x_\alpha \wedge u \leqslant y_\alpha.$$

for every $(\alpha, y) \geqslant (\alpha_0, 0)$. Therefore, $x_{(\alpha,y)} \overset{suo}{\to} 0$ in E. By hypothesis, $x_{(\alpha,y)} \overset{suo}{\to} 0$ in G. Since T is *suo*-continuous on G, $Tx_{(\alpha,y)} \overset{suo}{\to} 0$ in F. Let's take $w \in F_+$. There is a net $(v_{(\alpha,y)})$ in F such that $v_{(\alpha,y)} \downarrow 0$ and $Tx_{(\alpha,y)} \wedge w \leqslant v_{(\alpha,y)}$ for some (α_1, y_1) and for all $(\alpha, y) \geqslant (\alpha_1, y_1)$. Let $w_\alpha = v_{(\alpha,0)}$ for each α. It is not difficult to see that $0 \leqslant w_\alpha \downarrow 0$, since Γ doesn't have a maximal element. Let's choose $\alpha_2 > \alpha_1$. For each $\alpha \geqslant \alpha_2$ and for each $y \in G \cap [0, x_\alpha]$ we have $(\alpha, y) \geqslant (\alpha, 0) \geqslant (\alpha_1, y_1)$ and so,

$$Ty \wedge w = Tx_{(\alpha,y)} \wedge w \leqslant v_{(\alpha,y)} \leqslant v_{(\alpha,0)} = w_\alpha$$

Therefore, $\hat{T}x_\alpha \wedge w = S(x_\alpha) \wedge w \leqslant w_\alpha$ for each $\alpha \geqslant \alpha_2$. Thus, we obtain that $\hat{T}$ is *suo*-continuous. Since every *suo*-continuous operator is o-continuous and G is order dense in E, it is obtained that $\hat{T}$ is unique. $\rhd$

As a consequence of the preceding theorem, we obtain the following result.

Corollary 6. *If E is a suo-convergence reducing of E^δ, then $L_{uo}(E, F) = L_{suo}(E, F)$ for every Dedekind complete Riesz space F.*

$\lhd$ It is known that $L_{uo}(E, F) \subseteq L_{suo}(E, F)$. We have to prove that $L_{suo}(E, F) \subseteq L_{uo}(E, F)$. Let $T : E \to F$ be a *suo*-continuous operator. Since $L_{suo}(E, F)$ is an ideal in $L_b(E, F)$, it is enough to show that $|T|$ is a *uo*-continuous [2]. By Proposition 8, $|T|$ has a unique positive *suo*-continuous extension to all of E^δ. Let $|\hat{T}|$ denote the extension of $|T|$ to all of E^δ. Since E^δ and F are Dedekind complete Riesz spaces, $|\hat{T}| : E^\delta \to F$ is a *uo*-continuous. Then $|T|$ is a *uo*-continuous. $\rhd$

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Received December 6, 2024

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Владикавказский математический журнал
2025, Том 27, Выпуск 4, С. 148–156

ЧТО ОСТАЕТСЯ НЕИЗМЕННЫМ В ТИПАХ ПОРЯДКОВОЙ СХОДИМОСТИ?

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Аннотация. В данной статье мы исследуем какие свойства не зависят от того, рассматривается ли порядковая сходимость или неограниченная порядковая сходимость, а также неограниченная порядковая непрерывность или сильно неограниченная порядковая непрерывность. В [1] Гао и др. установили, что подрешетка пространства Рисса является порядково замкнутой тогда и только тогда, когда она является неограниченной порядково замкнутой. Показано, что σ-идеалы и неограниченные σ-идеалы — это одно и то же. Кроме того, установлено, что инъективные операторы, переводящие полосы на полосы, являются неограниченными порядково непрерывными, в то время как биективные порядково ограниченные сохраняющие дизъюнктность операторы также являются порядково непрерывными. Пусть G — порядково плотное мажорирующее подпространство Рисса пространства Рисса E, а F — дедекиндово

полное пространство Рисса. В [2] ставится вопрос: если $T : G \to F$ — положительный сильно неограниченно порядково непрерывный оператор, имеет ли T единственное положительное сильно неограниченное порядково непрерывное расширение на все E? Мы доказываем, что эта проблема имеет положительный ответ, если G наследует *suo*-сходимостью из E, а именно, если $x_\alpha \overset{suo}{\to} 0$ в E, то $x_\alpha \overset{uo}{\to} 0$ в G для любой сети (x_α) в G.

Ключевые слова: неограниченно порядково сходящийся, неограниченно порядково замкнутый идеал, неограниченно порядково непрерывный оператор, сильно неограниченно порядково непрерывный оператор.

AMS Subject Classification: 46A40, 47B65.

Образец цитирования: *Uyar A.* What Remains the Same in Order Convergence Types // Владикавк. мат. журн.—2025.—Т. 27, № 4.—С. 148–156 (in English). DOI: 10.46698/x9860-3651-6483-z.