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POSITIVE ISOMETRIES OF ORLICZ–KANTOROVICH SPACES

B. S. Zakirov[1] and V. I. Chilin[2]

[1] Tashkent State Transport University,
1 Temiryulchilar St., Tashkent 100167, Uzbekistan;
[2] National University of Uzbekistan,
Vuzgorodok, Tashkent 100174, Uzbekistan

E-mail: vladimirchil@gmail.com, botirzakirov@list.ru

Abstract. Let B be a complete Boolean algebra, $Q(B)$ the Stone compact of B, and let $C_\infty(Q(B))$ be the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. We consider the Orlicz–Kantorovich spaces $(L_\Phi(B,m), \|\cdot\|_\Phi) \subset C_\infty(Q(B))$ with the Luxembourg norm associated with an Orlicz function Φ and a vector-valued measure m, with values in the algebra of real-valued measurable functions. It is shown, that in the case when Φ satisfies the (Δ_2)-condition, the norm $\|\cdot\|_\Phi$ is order continuous, that is, $\|x_n\|_\Phi \downarrow \mathbf{0}$ for every sequence $\{x_n\} \subset L_\Phi(B,m)$ with $x_n \downarrow \mathbf{0}$. Moreover, in this case, the norm $\|\cdot\|_\Phi$ is strictly monotone, that is, the conditions $|x| \lneqq |y|$, $x, y \in L_\Phi(B,m)$, imply $\|x\|_\Phi \lneqq \|y\|_\Phi$. In addition, for positive elements $x, y \in L_\Phi(B,m)$, the equality $\|x+y\|_\Phi = \|x-y\|_\Phi$ is valid if and only if $x \cdot y = 0$. Using these properties of the Luxembourg norm, we prove that for any positive linear isometry $V : L_\Phi(B,m) \to L_\Phi(B,m)$ there exists an injective normal homomorphisms $T : C_\infty(Q(B)) \to C_\infty(Q(B))$ and a positive element $y \in L_\Phi(B,m)$ such that $V(x) = y \cdot T(x)$ for all $x \in L_\Phi(B,m)$.

Keywords: the Banach–Kantorovich space, the Orlicz function, vector-valued measure, positive isometry, normal homomorphism.

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1. Introduction

The development of the theory of integration for vector measures with values in complete vector lattices made it possible to construct useful examples of Banach–Kantorovich spaces. Important examples of such spaces include the "vector-valued" analogues of the L_p-spaces $L_p(B,m)$, $1 \leqslant p < \infty$ (see [1, 2]), and the Orlicz spaces $L_\Phi(B,m)$ [3–5] associated with a complete Boolean algebra B and the $L^0(\Omega)$-valued measure m, where $L^0(\Omega)$ is the algebra of real measurable functions on the measure space (Ω, Σ, μ) with a σ-finite numerical measure μ. If Ω is a singleton, then the class of Banach–Kantorovich spaces coincides with the class of real Banach spaces, important examples of which are symmetric spaces $E_{\mathbb{R}}(\Omega, \mathscr{A}, \mu)$ of real-valued measurable functions on (Ω, Σ, μ). The study of the geometric and topological properties of the spaces $E_{\mathbb{R}}(\Omega, \mathscr{A}, \mu)$ is firmly related to the problem of describing linear isometries of such spaces. The work on this problem began with the results of S. Banach [6], who gave a description of linear isometries for the spaces $L_p[0,1]$, $p \neq 2$. Later, Lamperti [7] described all linear isometries of the spaces $L_p(\Omega, \Sigma, \mu)$, $p \neq 2$, for any σ-finite measure spaces (Ω, Σ, μ). The approach in their proofs was to establish the property of preservation of disjointness for such isometries [8, Chapter 3].

To study surjective linear isometries on the broader class of functional symmetric spaces $E(\Omega, \mathscr{A}, \mu)$, different approaches are required, that depend on a scalar field. If $E_{\mathbb{C}}(\Omega, \mathscr{A}, \mu)$ is symmetric space of complex measurable functions on (Ω, Σ, μ), then G. Lumer's method [9] based on the theory of Hermitian operators can be effectively applied. For example, M. G. Zaidenberg [10, 11] used this method for description of all surjective linear isometries on the complex symmetric space $E_{\mathbb{C}}(\Omega, \mathscr{A}, \mu)$, where μ is a continuous measure. For the real symmetric space $E = E_{\mathbb{R}}([0,1], \mathscr{A}, \mu)$ of real-valued measurable functions on the segment $[0,1]$ with a Lebesgue measure μ, in the case when E is a separable space or has the Fatou property, a description of all surjective linear isometries on E was given by N. J. Kalton and B. Randrianantoanina [12]. They used methods of the theory of positive numerical operators. For real symmetric sequence spaces, a general form of surjective linear isometries was described by M. Sh. Braverman and E. M. Semenov [13, 14]. For complex separable symmetric sequence spaces (symmetric sequence spaces with the Fatou property), a general form of surjective linear isometries was described in [15] (respectively, in [16]).

However, the situation is more complicated in the case when isometries are not necessarily surjective. In this case, Y. Abramovich [17, Remark 2, p.78] emphasizes that, even positive isometries from a symmetric function space E into symmetric function space F may not necessarily have the "disjointness preserving" property. Still, in the commutative case, there exists an interesting and important special case when the latter property can be guaranteed for positive isometries "into". This special case was first considered in [18] and later reviewed and strengthened in [17, Corollary 6] (see also the proof of [19, Proposition 8]). The extra condition used in [17–19] is the requirement that the norm $\|\cdot\|_F$ on the Banach lattice F is strictly monotone, that is, $0 \leqslant x \lneqq y$, $x, y \in F$, implies that $\|x\|_F < \|y\|_F$. For the Orlicz function space $L_\Phi(\Omega, \Sigma, \mu)$ with an Orlicz function Φ satisfying the (Δ_2)-condition, the strict monotonicity of the norm allows to obtain a description of its positive isometries [20]. We also note that F. A. Sukochev and A. S. Veksler [21] introduced the property of K-strict monotonicity of the norm for non-commutative symmetric spaces, and used this property to give a description of positive isometries of non-commutative symmetric spaces.

In this paper, we show that for an Orlicz function Φ with the (Δ_2)-condition the norm on the Banach–Kantorovich space $L_\Phi(B, m)$ is strictly monotone. Using this property, we describe all positive isometries in $L_\Phi(B, m)$.

We use the terminology and notation of the theory of Boolean algebras from [22], the theory of vector lattices from [23], the theory of vector integration and the theory of Banach–Kantorovich spaces from [1].

2. Preliminaries

Let X be a real vector space, and let F be a complete vector lattice. Denote $F_+ = \{f \in F : f \geqslant 0\}$. The mapping $\|\cdot\| : X \to F_+$ is called an F-valued norm if for any $x, y \in X$, $\lambda \in \mathbb{R}$, the following properties hold: $\|x\| = 0 \Leftrightarrow x = 0$; $\|\lambda x\| = |\lambda|\,\|x\|$; $\|x + y\| \leqslant \|x\| + \|y\|$.

An F-valued norm $\|\cdot\|$ is said to be decomposable if for any $f_1, f_2 \in F_+$ and $x \in X$ with $\|x\| = f_1 + f_2$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\|x_i\| = f_i$, $i = 1, 2$.

A pair $(X, \|\cdot\|)$ with an F-valued norm is called a lattice normed space. If, in addition, the norm $\|\cdot\|$ is decomposable, then $(X, \|\cdot\|)$ is called decomposable.

We say that the net $\{x_\alpha\}_{\alpha\in A}$ from a lattice normed space $(X, \|\cdot\|)$ (bo)-converges to an element $x \in X$ (writing $x = (bo)\text{-}\lim x_\alpha$) if the net $(\|x - x_\alpha\|)_{\alpha\in A}$ (o)-converges to zero in the lattice F, that is, there exists a net $\{f_\alpha\}_{\alpha\in A} \subset F_+$ such that $f_\alpha \downarrow 0$, and $\|x - x_\alpha\| \leqslant f_\alpha$ for all $\alpha \in A$. A net $(x_\alpha)_{\alpha\in A} \subset X$ is called (bo)-fundamental if the net $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

A lattice normed space is called (bo)-complete if every (bo)-fundamental net in it (bo)-converges to an element of this space. A decomposable (bo)-complete lattice normed space is called a Banach–Kantorovich space.

An F-valued norm $\|\cdot\|$ on a vector lattice X is said to be monotonic if condition $|x| \leqslant |y|$, $x, y \in X$, implies that $\|x\| \leqslant \|y\|$. If a Banach–Kantorovich space $(X, \|\cdot\|_X)$ is a vector lattice and the norm $\|\cdot\|_X$ is monotonic, then it is called a Banach–Kantorovich lattice.

Let B be a complete Boolean algebra with zero $\mathbf{0}$ and unit $\mathbf{1}$. The exact upper and lower bounds of a set $\{e, q\} \subset B$ are denoted by $e \vee q$ and $e \wedge q$. A Boolean subalgebra A in B is called a regular if $\sup E \in A$, and $\inf E \in A$ for any subset $E \subset A$. Every regular Boolean subalgebra in B is a complete Boolean algebra.

A non-empty set E of nonzero elements from B is said to be disjoint if $e \wedge q = \mathbf{0}$ for any $e, q \in E$, $e \neq q$. A partition of unity in Boolean algebra is a disjoint family E in B such that $\sup E = \mathbf{1}$.

Let $Q(B)$ be the Stone compact of B, and let $L^0(B) := C_\infty(Q(B))$ be the commutative unital algebra over the field real numbers $\mathbb{R}$ of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$ (see, for example, [1, Chapter 1, Section 1.4.2], [23, Chapter V]). With respect to the partial order $x \leqslant y \Leftrightarrow y(t) - x(t) \geqslant 0$ for all $t \in Q(B) \setminus (x^{-1}(\pm\infty) \cup y^{-1}(\pm\infty))$, the algebra $L^0(B)$ is a complete vector lattice, and the set ∇ of all idempotents in $L^0(B)$ is a complete Boolean algebra with respect to the partial order induced by $L^0(B)$. In addition, ∇ is isomorphic to the Boolean algebra B. It is known that the set $C(B) := C(Q(B))$ of all continuous real-valued functions on $Q(B)$ is a subalgebra in $L^0(B)$, and $C(B)$ is a Banach space with respect to the uniform norm $\|x\|_\infty = \sup_{t\in Q(B)} |x(t)|$.

We denote by $s(x) := \sup_{n\geqslant 1} \{|x| > n^{-1}\}$, the support of an element $x \in L^0(B)$, where $\chi_{E_\lambda} = \{|x| > \lambda\} \in B$ is the characteristic function of the set E_λ which is the closure in $Q(B)$ of the set $\{t \in Q(B) : |x(t)| > \lambda\}$, $\lambda \geqslant 0$.

For any nonzero $x \in L^0(B)$ define $i(x)$ as the inverse element to x on its support, i. e.,

$$i(x)(t) = \begin{cases} \dfrac{1}{x(t)}, & \text{if } x(t) \neq 0, \\ 0, & \text{if } x(t) = 0. \end{cases}$$

It is clear that $i(x) \in L^0(B)$ and $i(x) \cdot x = s(x)$.

Let (Ω, Σ, μ) be a σ-finite measure space, and let $L^0(\Omega)$ be the algebra of equivalence classes of almost everywhere finite real-valued measurable functions on Ω. With respect to the partial order $f \leqslant g \Leftrightarrow g - f \geqslant 0$ (almost everywhere), the algebra $L^0(\Omega)$ is a complete vector lattice, and the set $B(\Omega)$ of all idempotents in $L^0(\Omega)$ is a complete Boolean algebra with respect to the partial order induced by $L^0(\Omega)$. Since μ is a σ-finite measure, it follows that $B(\Omega)$ is a Boolean algebra of countable type, that is, any subset $E \subset B(\Omega)$ of non-zero pairwise disjoint elements is at most countable. Thus, for any increasing net $x_\alpha \uparrow x \in L^0(\Omega)$, $\{x_\alpha\}_{\alpha\in A} \subset L^0(\Omega)$, there exists a sequence $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n \leqslant \ldots$ such that $x_{\alpha_n} \uparrow x$ (see, for example, [23, Chapter VI, § 2]).

A mapping $m : B \to L^0(\Omega)$ is called a $L^0(\Omega)$-*valued measure* if it satisfies the following conditions:

1) $m(e) \geqslant 0$ for all $e \in B$;
2) $m(e \vee g) = m(e) + m(g)$ for any $e, g \in B$ with $e \wedge g = \mathbf{0}$;
3) $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow \mathbf{0}$, $\{e_\alpha\} \subset B$.

A measure m is said to be strictly positive if $m(e) = 0$ implies $e = \mathbf{0}$. In this case, B is a Boolean algebra of countable type, thus, in condition 3) above, instead of the net $e_\alpha \downarrow 0$, one can take a sequence $e_n \downarrow 0$, $\{e_n\}_{n=1}^\infty \subset B$.

A strictly positive $L^0(\Omega)$-valued measure m is said to be *decomposable* if for any $e \in B$ and a decomposition $m(e) = f_1 + f_2$, $f_1, f_2 \in L^0(\Omega)_+$, there exist $e_1, e_2 \in B$, such that $e = e_1 \vee e_2$, and $m(e_i) = f_i$, $i = 1, 2$. A measure m is decomposable if and only if it is a Maharam measure, that is, for any $e \in B$, $0 \leqslant f \leqslant m(e)$, $f \in L^0(\Omega)$, there exists $q \in B$, $q \leqslant e$, such that $m(q) = f$ [24]. Maharam measures have the following important property.

Proposition 1 [24, Proposition 3.2]**.** *For each $L^0(\Omega)$-valued Maharam measure $m : B \to L^0(\Omega)$ there exists a unique injective completely additive homomorphism $\varphi : B(\Omega) \to B$ such that $\varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$.*

Let $m : B \to L^0(\Omega)$ be a Maharam measure. We identify B with the Boolean algebra of idempotents in $L^0(B)$, i. e., we assume that $B \subset L^0(B)$. By Proposition 1, there exists a regular Boolean subalgebra $\nabla(m)$ in B and an isomorphism φ from $B(\Omega)$ onto $\nabla(m)$ such that $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$. In this case, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$ (the corresponding isomorphism will also be denoted by φ). Thus, the algebra $C_\infty(Q(\nabla(m)))$ can be considered as a subalgebra and as a regular vector sublattice of $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$). In particular, $L^0(B)$ is an $L^0(\nabla(m))$-module.

Consider the vector sublattice $S(B)$ in $L^0(B)$ of all simple elements $x = \sum_{i=1}^n \alpha_i e_i$, where $\alpha_i \in \mathbb{R}$, $e_i \in B$, $e_i \cdot e_j = \mathbf{0}$, $i, j = 1, \dots, n$. The formula

$$I_m(x) := \int x\, dm := \sum_{i=1}^n \alpha_i m(e_i), \quad x \in S(B),$$

correctly defines the linear operator $I_m : S(B) \to L^0(\Omega)$.

A positive element $x \in L^0(B)_+$ is called *m-integrable*, if there exists a sequence $\{x_n\}_{n=1}^\infty \subset S(B)$, $0 \leqslant x_n \uparrow x$, such that there is a supremum $\sup_{n\geqslant 1} I_m(x_n)$ in the lattice $L^0(\Omega)$. In this case, the integral of the element x with respect to the measure m is defined by

$$I_m(x) := \int x\, dm := (o)\text{-}\lim_{n\to\infty} \int x_n\, dm.$$

It is known that the definition of the integral $I_m(x)$ does not depend on the choice of the sequence $\{x_n\}_{n=1}^\infty \subset S(B)$, $0 \leqslant x_n \uparrow x$, for which there exists $\sup_{n\geqslant 1} I_m(x_n)$ (see, for example, [1, 6.1.3]).

An element $x \in L^0(B)$ is called *m-integrable* if its positive and negative parts x_+ and x_- are m-integrable. The set of all m-integrable elements is denoted by $L^1(B, m)$, and for every $x \in L^1(B, m)$ we have

$$\int x\, dm := \int x_+\, dm - \int x_-\, dm.$$

If $\|x\|_{1,m} := \int |x| dm$, $x \in L^1(B, m)$, then the pair $(L^1(B, m), \|x\|_{1,m})$ is a lattice normed space over $L^0(\Omega)$ [1, 6.1.3]. Moreover, in the case when $m : B \to L^0(\Omega)$ is a Maharam measure, the pair $(L^1(B,m), \|x\|_{1,m})$ is a Banach–Kantorovich space. In addition,

$$L^0(\nabla(m)) \cdot L^1(B,m) \subset L^1(B,m), \quad \int (\varphi(\alpha)x)\, dm = \alpha \int x\, dm, \quad \left| \int x\, dm \right| \leqslant \int |x|\, dm$$

for all $x \in L^1(B,m)$, $\alpha \in L^0(\Omega)$ [1, Theorem 6.1.10].

Let $p \in [1,\infty)$, and let

$$L^p(B,m) = \{x \in L^0(B) : |x|^p \in L^1(B,m)\},$$

$$\|x\|_{p,m} := \left[\int |x|^p\, dm \right]^{\frac{1}{p}}, \quad x \in L^p(B,m).$$

It is known that for the Maharam measure m the pair $(L^p(B,m), \|x\|_{p,m})$ is a Banach–Kantorovich space [2, 4.2.2]. In addition,

$$\varphi(\alpha)x \in L^p(B,m) \quad \forall\, x \in L^p(B,m),\ \alpha \in L^0(\Omega),\ 1 \leqslant p < \infty,$$

and $\|\varphi(\alpha)x\|_{p,m} = |\alpha|\|x\|_{p,m}$.

3. Orlicz–Kantorovich Lattices for L^0-Valued Measures

Let B be a complete Boolean algebra, and let $m : B \to L^0(\Omega)$ be a Maharam measure for which $m(\mathbf{1}_B) = \mathbf{1}_{B(\Omega)}$. The algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$, which is a subalgebra and a regular vector sublattice of $L^0(B) = C_\infty(Q(B))$.

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if Φ is left continuous, convex, increasing function such that $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t > 0$. It is known that the derivative Φ' exists almost everywhere on $(0,\infty)$, and there is a unique increasing left-continuous function $\phi : [0,\infty) \to [0,\infty)$, such that $\phi = \Phi'$ almost everywhere on $[0,\infty)$ and

$$\Phi(t) = \int_0^t \phi(u)\, du \quad (\forall\, t > 0).$$

In particular, $\Phi(t) < \Phi(s)$ for all $0 < t < s$ (see, for example, [25, Chapter 13, § 13.1]).

For each function $x \in L^0(B)$ the set $G = \{t \in Q(B) : -\infty < x(t) < +\infty\}$ is everywhere dense and open in $Q(B)$. Therefore, for a continuous function $\Phi(x(t))$, $t \in G$, there is a unique continuous extension $y(t)$ to $Q(B)$ (see [23] , Lemma V.2.1), i. e., $\Phi(x) := y \in L^0(B)$. It is clear that

$$\Phi(ex) = e\Phi(x) \quad (\forall\, e \in B).$$

Moreover, since Φ is a convex function on $[0,\infty)$ and $\Phi(0) = 0$, it follows that

$$y \cdot \Phi(|x|) \geqslant \Phi(|y \cdot x|) \quad (\forall\, x \in L^0(B),\ y \in L^0(B)) \ \text{ with } \ \mathbf{0} \leqslant y \leqslant \mathbf{1}. \tag{1}$$

In addition, as the function Φ is increasing and left continuity, we obtain the following

Lemma 1. *If $x_n, x \in L^0(B)_+$ and $x_n \downarrow x$, then $\Phi(x_n) \downarrow \Phi(x)$.*

Following the traditional scheme (see, for example, [26, Chapter 2]), we introduce the Orlicz classes and Orlicz spaces associated with an $L^0(\Omega)$-valued measure m and an Orlicz function Φ. Let $L^0(\Omega)_{++}$ be the set of all positive elements $\lambda \in L^0_+(\Omega)$ such that $s(\lambda) = \mathbf{1}$. It is clear that for any $\lambda \in L^0(\Omega)_{++}$ there exists $\lambda^{-1} \in L^0(\Omega)_{++}$ such that $\lambda \cdot \lambda^{-1} = \mathbf{1}$.

As in [26, Chapter 2]) we define the *Young class*

$$Y_\Phi := Y_\Phi(B,m) = \big\{x \in L^0(B) : \Phi(|x|) \in L^1(B,m)\big\}$$

and the *Orlicz space*

$$L_\Phi := L_\Phi(B,m) = \big\{x \in L^0(B) : \Phi\big(\lambda^{-1}|x|\big) \in L^1(B,m) \text{ for some } \lambda \in L^0(\Omega)_{++}\big\}.$$

Let

$$H_\Phi := H_\Phi(B,m) = \big\{x \in L^0(B) : \Phi\big(\lambda^{-1}|x|\big) \in L^1(B,m) \text{ for all } \lambda \in L^0(\Omega)_{++}\big\}.$$

If Ω is a singleton, then the above definitions coincide with the well-known definitions of the Young class and the Orlicz space of measurable functions (see, for example, [26, Chapter 2]). It is clear that

$$H_\Phi(B,m) \subset Y_\Phi(B,m) \subset L_\Phi(B,m).$$

In addition, H_Φ is a linear subspace of the linear space L_Φ, and Y_Φ is an L^0-convex subset of L_Φ, that is, $\lambda Y_\Phi + (\mathbf{1}-\lambda)Y_\Phi \subset Y_\Phi$ for all $\lambda \in L^0(\Omega)$, $\mathbf{0} \leqslant \lambda \leqslant \mathbf{1}$ (see (1)); however, Y_Φ may not be the same as L_Φ. As in the case of classical Orlicz spaces, it is established that $H_\Phi = L_\Phi$ if and only if $Y_\Phi = 2Y_\Phi$ (see, for example, [26, Chapter 2, Proposition 2.1.15]). Note also that if $\Phi(t) = t^p/p$, $t \geqslant 0$, $1 \leqslant p < \infty$, we have $L_\Phi(B,m) = L^p(B,m)$.

Let

$$\mathscr{I}_\Phi(x) = \int \Phi(|x|)\, dm, \quad \text{where } x \in Y_\Phi(B,m).$$

It is clear that

$$\mathscr{I}_\Phi(ex) = e\mathscr{I}_\Phi(x) \quad (\forall\, e \in B).$$

In addition, using (1), we obtain for any $\mathbf{1} \leqslant \lambda \in L^0(\Omega)$

$$\lambda^{-1}\mathscr{I}_\Phi(x) = \int \lambda^{-1}\Phi(|x|)\, dm \geqslant \int \Phi\big(\big|\lambda^{-1}x\big|\big)\, dm = \mathscr{I}_\Phi\big(\lambda^{-1}x\big). \tag{2}$$

Define an $L^0(\Omega)$-valued Luxembourg norm on $L_\Phi(B,m)$, setting

$$\|x\|_\Phi := \inf\big\{\lambda \in L^0(\Omega)_{++} : \mathscr{I}_\Phi(\lambda^{-1}x) \leqslant \mathbf{1}\big\}, \quad x \in L_\Phi(B,m).$$

It is known that the pair $(L_\Phi(B,m), \|\cdot\|_\Phi)$ is a Banach–Kantorovich lattice, called the *Orlicz–Kantorovich lattice associated with $L^0(\Omega)$-valued measure* [3, 4]. Moreover, the norm $\|\cdot\|_\Phi$ has the following important property

$$\|\alpha x\|_\Phi = |\alpha|\|x\|_\Phi \quad (\forall\, x \in L_\Phi(B,m),\ \alpha \in L^0(\Omega))$$

(see [3, Proposition 2.7]). We also note that it follows from $m(\mathbf{1}) = \mathbf{1}$ that $\mathbf{1} \in L_\Phi(B,m)$, and therefore $C(B) \subset L_\Phi(B,m)$. In addition, for any $x \in C(B)$ we get that

$$\|x\|_\Phi \leqslant \big\|\|x\|_\infty \cdot \mathbf{1}\big\|_\Phi = \|x\|_\infty \cdot \|\mathbf{1}\|_\Phi,$$

in particular, $\|x_n - x\|_\infty \to 0$, $x_n, x \in C(B) \Longrightarrow \|x_n - x\|_\Phi \to 0$.

We need the following inequalities.

Proposition 2. *If $\mathbf{0} \neq x \in L_\Phi(B,m)$, then $s(x) \leqslant s(\|x\|_\Phi)$ and $\mathscr{I}_\Phi(i(\|x\|_\Phi)x) \leqslant s(\|x\|_\Phi)$.*

$\lhd$ Since

$$\big\|x(\mathbf{1} - s(\|x\|_\Phi))\big\|_\Phi = \big(\mathbf{1} - s(\|x\|_\Phi)\big)\|x\|_\Phi = \mathbf{0},$$

it follows that $x(\mathbf{1} - s(\|x\|_\Phi)) = \mathbf{0}$. Thus $s(x) \leqslant s(\|x\|_\Phi)$.

Let's show now that $\mathscr{I}_\Phi(i(\|x\|_\Phi)x) \leqslant s(\|x\|_\Phi)$. Denote

$$\Lambda(x) = \left\{\lambda \in L^0(\Omega)_{++} : \mathscr{I}_\Phi(\lambda^{-1}x) \leqslant \mathbf{1}\right\}$$

and let $\lambda_1, \lambda_2 \in \Lambda(x)$. Then

$$q = \left\{\lambda_1 \leqslant \lambda_2\right\} \in B(\Omega) \quad \text{and} \quad \gamma = \lambda_1 \wedge \lambda_2 = \lambda_1 q + \lambda_2(\mathbf{1} - q) \in L^0(\Omega)_{++},$$

in addition,

$$\mathscr{I}_\Phi\big(\gamma^{-1}x\big) = \mathscr{I}_\Phi\big(\big(\lambda_1^{-1}q + \lambda_2^{-1}(\mathbf{1}-q)\big)x\big) = q\mathscr{I}_\Phi\big(\lambda_1^{-1}x\big) + (\mathbf{1}-q)\mathscr{I}_\Phi\big(\lambda_2^{-1}x\big) \leqslant q + (\mathbf{1}-q) = \mathbf{1},$$

that is, $\gamma \in \Lambda(x)$. Using mathematical induction, we get that

$$\inf_{1 \leqslant i \leqslant n} \lambda_i \in \Lambda(x) \ \text{ for any finite subset } \{\lambda_1, \dots, \lambda_n\} \subset \Lambda(x).$$

Since $B(\Omega)$ is a Boolean algebra of countable type, it follows that there exists a sequence $\{\lambda_n\} \subset \Lambda(x)$ such that

$$\lambda_n \downarrow \inf \Lambda(x) = \|x\|_\Phi.$$

If $e = s(\|x\|_\Phi)$, then $e \in L^0(\Omega)$ and

$$\lambda_n e \downarrow e\|x\|_\Phi \implies \lambda_n^{-1} e \uparrow e \cdot i(\|x\|_\Phi).$$

Since $\mathscr{I}_\Phi(\lambda_n^{-1}x) \leqslant \mathbf{1}$, it follows that $\mathscr{I}_\Phi((\lambda_n^{-1}e)x) \leqslant e$. Using now the theorem of monotone convergence [1, Chapter VI, Theorem 6.1.4], we obtain that

$$\mathscr{I}_\Phi(i(\|x\|_\Phi)x) \leqslant e = s(\|x\|_\Phi). \quad \rhd$$

In the following proposition, we use the inequality $\alpha \lneqq \beta$, $\alpha, \beta \in L^0(\Omega)$, which means that $\alpha \leqslant \beta$ and $\alpha \neq \beta$.

Proposition 3. *Let $x \in L_\Phi(B, m)$ and $\mathbf{0} \neq e \in B(\Omega)$. Then*

(i) the inequality $\|ex\|_\Phi \leqslant e$ (resp., $\|ex\|_\Phi \lneqq e$) implies the inequality $\mathscr{I}_\Phi(ex) \leqslant e$ (resp., $\mathscr{I}_\Phi(ex) \lneqq e$);

(ii) the inequality $\|ex\|_\Phi \gneqq e$ implies the inequality $\mathscr{I}_\Phi(ex) \gneqq e$;

(iii) if $x \in H_\Phi$, then $\|ex\|_\Phi = e$ if and only if $\mathscr{I}_\Phi(ex) = e$.

$\lhd$ (*i*) Let $x \in L_\Phi(B, m)$, $e \in B(\Omega)$, $\mathbf{0} \neq \|ex\|_\Phi \leqslant e$ and $g = s(\|ex\|_\Phi)$. Then $g \leqslant e$, $g \leqslant i(\|ex\|_\Phi)$ and, using Proposition 2, we obtain that

$$\mathscr{I}_\Phi(gex) \leqslant \mathscr{I}_\Phi(i(\|ex\|_\Phi)ex) \leqslant g.$$

Moreover, from the equality $\|ex - egx\|_\Phi = (e - eg)\|ex\|_\Phi = \mathbf{0}$, we get that $ex = egx$. Thus

$$\mathscr{I}_\Phi(ex) = \mathscr{I}_\Phi(egx) \leqslant g \leqslant e.$$

Let now $\|ex\|_\Phi \lneqq e$, that is, $e \neq \|ex\|_\Phi \leqslant e$. If $s(\|ex\|_\Phi) = p < e$, then $\|ex\|_\Phi \leqslant p$ and $s(ex) \leqslant s(\|ex\|_\Phi) = p$, in particular, $px = pex$. From what was proved above, we get

$$\mathscr{I}_\Phi(ex) = \mathscr{I}_\Phi(pex) = \mathscr{I}_\Phi(px) \leqslant p < e.$$

If $s(\|ex\|_\Phi) = e$, then there exist

$$\varepsilon \in (0,1), \ \mathbf{0} \neq q \leqslant s(\|ex\|_\Phi) = g, \ q \in B(\Omega), \ \text{ such that } \ q\|ex\|_\Phi \leqslant (1-\varepsilon)q.$$

Thus, $qi(\|ex\|_\Phi) \geqslant \frac{q}{1-\varepsilon}$. Using Proposition 2, we get

$$\frac{\mathscr{I}_\Phi(qex)}{1-\varepsilon} \leqslant \mathscr{I}_\Phi(qi(\|ex\|_\Phi)ex) = q\mathscr{I}_\Phi(i(\|ex\|_\Phi)ex) \leqslant qg = q.$$

If $\mathscr{I}_\Phi(ex) = e$, then $q\mathscr{I}_\Phi(ex) = q$ and

$$q \geqslant \frac{\mathscr{I}_\Phi(qex)}{1-\varepsilon} = \frac{q}{1-\varepsilon},$$

which is wrong. Thus $\mathscr{I}_\Phi(ex) \lneqq e$.

(ii) Let $x \in L_\Phi(B,m)$, $e \in B(\Omega)$ and $\|ex\|_\Phi \gneqq e$. Let us show that for any $\mathbf{0} \neq p \leqslant e$, $p \in B(\Omega)$ there exists $\mathbf{0} \neq q \leqslant p$, $q \in B(\Omega)$ such that $\mathscr{I}_\Phi(qx) \gneqq q$.

Choose $\alpha \in L^0(\Omega)_+$ such that $p \lneqq \alpha \lneqq \|px\|_\Phi$. It is clear that $s(\alpha) = p$ and $p \gneqq i(\alpha)$. Since $\alpha \lneqq \|px\|_\Phi$ and

$$\begin{aligned} \|px\|_\Phi = p\|x\|_\Phi &= \inf\big\{p\lambda :\ \lambda \in L^0(\Omega)_{++},\ \mathscr{I}_\Phi(\lambda^{-1}px) \leqslant \mathbf{1}\big\} \\ &= \inf\big\{\gamma :\ \gamma \in p\,L^0(\Omega)_{++},\ \mathscr{I}_\Phi(i(\gamma)px) \leqslant p\big\}, \end{aligned} \tag{3}$$

it follows that $\mathscr{I}_\Phi(i(\alpha)px) \nleqslant p$. Thus, there exists $\mathbf{0} \neq q \leqslant p$, $q \in B(\Omega)$ such that

$$\mathscr{I}_\Phi(i(\alpha)qx) = q\mathscr{I}_\Phi(i(\alpha)px) \gneqq q.$$

Using the inequalities $i(\alpha) \lneqq p$ and (2), we obtain

$$i(\alpha)\mathscr{I}_\Phi(qx) \geqslant \mathscr{I}_\Phi(i(\alpha)qx) \gneqq q, \quad \text{e. a.,} \quad \mathscr{I}_\Phi(qx) \gneqq \alpha q \gneqq q.$$

Using now the "Principle of Exhaustion" for complete Boolean algebras [22, Chapter III, § 2], we get that there exists a disjoint set $\{q_i\}_{i\in I} \subset B(\Omega)$ such that

$$\sup_{i\in I} q_i = e \ \text{ and } \ \mathscr{I}_\Phi(q_i x) \gneqq q_i \quad (\forall\, i \in I).$$

Consequently, $\mathscr{I}_\Phi(ex) \gneqq e$.

(iii) Let $x \in H_\Phi$, $\mathbf{0} \neq e \in B(\Omega)$, and let $\|ex\|_\Phi = e$. Then by the part *(i)*, we get that $\mathscr{I}_\Phi(ex) \leqslant e$. If $\mathscr{I}_\Phi(ex) \neq e$, then there exist $\varepsilon \in (0,1)$ and $\mathbf{0} \neq p < e$, $p \in B(\Omega)$ such that

$$\mathscr{I}_\Phi(px) < (1-\varepsilon)p \implies \mathscr{I}_\Phi\left(\frac{1}{1-\varepsilon}px\right) \lneqq p.$$

Using (3), we get that $\|px\|_\Phi \leqslant 1-\varepsilon$. Thus

$$p = pe = p\|ex\|_\Phi = \|px\|_\Phi \leqslant (1-\varepsilon)p,$$

which is impossible. Consequently, $\mathscr{I}_\Phi(ex) = e$.

Let now $\mathscr{I}_\Phi(ex) = e$. Suppose that $\|ex\|_\Phi \nleqslant e$. Since $s(\|ex\|_\Phi) \leqslant e$, it follows that there exist $\delta \in (0,1)$ and $\mathbf{0} \neq q < e$, $q \in B(\Omega)$ such that

$$\|qx\|_\Phi = q\|ex\|_\Phi \gneqq (1+\delta)q \gneqq q.$$

Thus, $\mathscr{I}_\Phi(qx) \gneqq q$ (see *(ii)*). Therefore, $q = qe = q\mathscr{I}_\Phi(ex) = \mathscr{I}_\Phi(qx) \gneqq q$. From this contradiction it follows that $\|ex\|_\Phi \leqslant e$. If $\|ex\|_\Phi \lneqq e$, then by the part *(i)*, we get that $e = \mathscr{I}_\Phi(ex) \lneqq e$, which is impossible. Thus $\|ex\|_\Phi = e$. $\rhd$

Definition 1. An Orlicz function Φ is said to satisfy the (Δ_2)-*condition* if $0 < \Phi(t) < \infty$ for all $t > 0$ and $\sup_{0<t<\infty} \frac{\Phi(2t)}{\Phi(t)} < \infty$.

Repeating the proof of Theorem 2.1.17 (1) [26, Chapter 2], we obtain the following version of it for the Orlicz–Kantorovich modules $L_\Phi(B,m)$.

Proposition 4. *If an Orlicz function Φ satisfies the (Δ_2)-condition, then $L_\Phi = H_\Phi$.*

Using now Propositions 3 (*iii*) and 4, we get the following

Proposition 5. *If an Orlicz function Φ satisfies the (Δ_2)-condition and $x \in L_\Phi(B,m)$, $\mathbf{0} \neq e \in B(\Omega)$, then $\|ex\|_\Phi = e$ if and only if $\mathscr{I}_\Phi(ex) = e$.*

We say that the Luxembourg norm $\|\cdot\|_\Phi$ is order continuous if $\|x_n\|_\Phi \downarrow \mathbf{0}$ for every sequence $\{x_n\} \subset L_\Phi$ with $x_n \downarrow \mathbf{0}$. It is clear that in this case, for any sequence $\{x_n\} \subset L_\Phi$ with $x_n \uparrow x \in L_\Phi$ we have that $\|x - x_n\|_\Phi \downarrow \mathbf{0}$.

Proposition 6. *If an Orlicz function Φ satisfies the (Δ_2)-condition, then the Luxembourg norm $\|\cdot\|_\Phi$ is order continuous.*

$\lhd$ Let $\{x_n\} \subset L_\Phi$ and $x_n \downarrow \mathbf{0}$. By Lemma 1, we have $\Phi(x_n) \downarrow \mathbf{0}$. Since $\mathbf{0} \leqslant x_n \in L_\Phi = H_\Phi$ (see Proposition 4), it follows that $x_n \in L^1(B,m)$ for all n, and using the convergence $x_n \downarrow \mathbf{0}$, we obtain that $\mathscr{I}_\Phi(x_n) \downarrow \mathbf{0}$.

Let's show that $\|x_n\|_\Phi \downarrow \mathbf{0}$. Suppose that $\|x_n\|_\Phi \downarrow \alpha \neq \mathbf{0}$. Then there exist $\varepsilon > 0$ and $\mathbf{0} \neq e \in B(\Omega)$ such that $\|ex_n\|_\Phi = e\|x_n\|_\Phi \geqslant e\alpha \gneqq \varepsilon e$, that is, $\|e(\varepsilon^{-1} \cdot x_n)\|_\Phi \gneqq e$ for all $n = 1, 2, \ldots$ Using now Proposition 3 (*ii*), we obtain $\varepsilon^{-1}\mathscr{I}_\Phi(ex_n)) = \mathscr{I}_\Phi(e(\varepsilon^{-1} \cdot x_n)) \gneqq e$, that is, $e\mathscr{I}_\Phi(x_n)) = \mathscr{I}_\Phi(ex_n)) \gneqq \varepsilon \cdot e$, $n = 1, 2, \ldots$, which contradicts the convergence $\mathscr{I}_\Phi(x_n) \downarrow \mathbf{0}$. $\rhd$

4. Positive Linear Isometries in Orlicz–Kantorovich Spaces

We say that the norm $\|\cdot\|_\Phi$ is strictly monotone if the conditions $|x| \lneqq |y|$ $x, y \in L_\Phi(B,m)$ imply $\|x\|_\Phi \lneqq \|y\|_\Phi$.

Using Proposition 5 we obtain the following

Proposition 7. *If an Orlicz function Φ satisfies the (Δ_2)-condition, then the Luxembourg norm $\|\cdot\|_\Phi$ is strictly monotone.*

$\lhd$ Let $x, y \in L_\Phi(B,m)$, $\mathbf{0} \neq |x| \lneqq |y|$, and let $\alpha = \|y\|_\Phi$. Since $s(y) \leqslant s(\|y\|_\Phi)$ (see Proposition 2), it follows that

$$\big\|s(\|y\|_\Phi)i(\alpha)y\big\|_\Phi = i(\alpha)\|y\|_\Phi = s(\|y\|_\Phi) \in B(\Omega).$$

Using Proposition 5, we get that $s(\|y\|_\Phi) = \mathscr{I}_\Phi(i(\alpha)y)$. Since the function Φ is strictly increasing and $|x| \lneqq |y|$, it follows that

$$\mathscr{I}_\Phi\left(i(\alpha)x\right) = \int \Phi((i(\alpha)|x|)\, dm \lneqq \int \Phi\left(i(\alpha)|y|\right) dm = \mathscr{I}_\Phi\left(i(\alpha)y\right). \tag{3}$$

If $\|x\|_\Phi = \|y\|_\Phi$, then

$$\big\|s(\|y\|_\Phi)i(\alpha)x)\big\|_\Phi = s(\|y\|_\Phi)i(\alpha)\|x\|_\Phi = s(\|y\|_\Phi)i(\alpha)\|y\|_\Phi = s(\|y\|_\Phi),$$

and, by Proposition 5, we get

$$\mathscr{I}_\Phi\left(i(\alpha)x\right) = \mathscr{I}_\Phi\left(s(\|y\|_\Phi)i(\alpha)x\right) = s(\|y\|_\Phi),$$

which contradicts the inequality (3). Consequently, $\|x\|_\Phi \neq \|y\|_\Phi$. $\rhd$

Corollary 1. *If an Orlicz function Φ satisfies the (Δ_2)-condition, and $x, y \in L_\Phi(B, m)_+$ then $x \cdot y = \mathbf{0}$ if and only $\|x + y\|_\Phi = \|x - y\|_\Phi$.*

$\lhd$ If $x \neq \mathbf{0}$, $y \neq \mathbf{0}$ and $x \cdot y = \mathbf{0}$, then $|x + y| = |x - y|$ and $\|x + y\|_\Phi = \|x - y\|_\Phi$.

Conversely, let $x, y \in L_\Phi(B, m)_+$ and $\|x + y\|_\Phi = \|x - y\|_\Phi$. Since Φ satisfies the Δ_2-condition, it follows that the norm $\|\cdot\|_\Phi$ is strictly monotone (see Proposition 7). If $x \cdot y \neq \mathbf{0}$, then $|x - y| \lneqq |x + y|$. Thus $\|x + y\|_\Phi \neq \|x - y\|_\Phi$, which is not true. Consequently, $x \cdot y = \mathbf{0}$. $\rhd$

Recall that a linear operator T in a vector lattice X is called positive if $T(x) \geqslant \mathbf{0}$ for all $\mathbf{0} \leqslant x \in X$. For any positive operator T we have $|T(x)| \leqslant T(|x|)$, where $|x| = x_+ + x_-$, and x_+, x_- are the positive and negative parts of an element $x \in X$ [1, Chapter 3, Section 3.1.1].

Corollary 2. *Let the Orlicz function Φ satisfy the (Δ_2)-condition, and let $V : L_\Phi(B, m) \to L_\Phi(B, m)$ be a positive linear isometry. If $x, y \in L_\Phi(B, m)_+$ and $x \cdot y = \mathbf{0}$, then $V(x) \cdot V(y) = \mathbf{0}$.*

$\lhd$ If $x, y \in L_\Phi(B, m)_+$ and $x \cdot y = \mathbf{0}$, then $|x + y| = |x - y|$ and

$$\begin{aligned} \|V(x) + V(y)\|_\Phi &= \|V(x + y)\|_\Phi = \|x + y\|_\Phi = \|\, |x - y| \,\|_\Phi \\ &= \|x - y\|_\Phi = \|V(x - y)\|_\Phi = \|V(x) - V(y)\|_\Phi. \end{aligned}$$

Since $V(x) \geqslant 0$, $V(y) \geqslant 0$, it follows that $V(x) \cdot V(y) = 0$ (see Corollary 1). $\rhd$

A linear operator $T : L^0(B) \to L^0(B)$ is called a homomorphism if $T(x \cdot y) = T(x) \cdot T(y)$ for all $x, y \in L^0(B)$. It is clear that any homomorphism $T : L^0(B) \to L^0(B)$ is a positive operator.

A positive linear operator $T : L^0(B) \to L^0(B)$ is called *normal* (resp., *completely additive*) if $T(\sup_\alpha x_\alpha) = \sup_\alpha T(x_\alpha)$ for any increasing net $\{x_\alpha\} \subset L^0(B)$, such that $\mathbf{0} \leqslant x_\alpha \uparrow x \in L^0(B)$ (resp., $T\big(\sum_{i \in I} e_i\big) = \sup_{\alpha \in A} \sum_{i \in \alpha} T(e_i)$, for every family of idempotents $\{e_i\}_{i \in I} \subset B$, $e_i e_j = \mathbf{0}$, $i \neq j$, $i, j \in I$, where $A = \{\alpha\}$ is the directed poset of all finite subsets of I, ordered by inclusion).

It is clear that the normality property for a positive linear operator implies that this operator is a completely additive one. In the case when $T : L^0(B) \to L^0(B)$ is a homomorphism, the inverse implication is also valid, that is, every completely additive homomorphism of $T : L^0(B) \to L^0(B)$ is a normal operator [27, Theorem 4].

Since $m(\mathbf{1}) = \mathbf{1}$ and μ is a σ-finite measure, it follows that there exists a sequence $\{e_n\} \subset B(\Omega)$ such that

$$\mu(e_n) < \infty, \ \ e_n e_k = \mathbf{0}, \ \ n \neq k, \ \ n, k = 1, 2, \ldots, \ \ \sup_{n \geqslant 1} e_n = \mathbf{1},$$

and $\{e_n \cdot m(q) : q \in B\} \subset L^1(\Omega, \Sigma, \mu)$. Thus the function $\nu(q) = \sum_{n=1}^\infty \int_\Omega e_n m(q) d\mu$ is a σ-finite numerical measure on the Boolean algebra B, in particular, B is a Boolean algebra of countable type. In this case, in the definition of normality (resp., completely additivity) of a positive linear operator $T : L^0(B) \to L^0(B)$, instead of an increasing net $\{x_\alpha\} \subset L^0(B)$ (resp., a family of idempotents $\{e_i\}_{i \in I} \subset B$, $e_i e_j = \mathbf{0}$, $i \neq j$, $i, j \in I$), one should take a sequence $\{x_n\} \subset L^0(B)$ (resp., a countable family of idempotents $\{e_i\}_{i \geqslant 1} \subset B$, $e_i e_j = \mathbf{0}$, $i \neq j$, $i, j = 1, 2 \ldots$).

The following theorem gives a description of all positive linear isometries acting in an Orlicz–Kantorovich spaces.

Theorem 1. *Let an Orlicz function Φ satisfy the (Δ_2)-condition, and let $V : L_\Phi(B, m) \to L_\Phi(B, m)$ be a positive linear isometry. Then there exist an injective normal homomorphism $T : L^0(B) \to L^0(B)$ and a positive element $y \in L_\Phi(B, m)$ such that $V(x) = y \cdot T(x)$ for all $x \in L_\Phi(B, m)$.*

$\lhd$ Define the mapping $\varphi : B \to B$, by setting $\varphi(e) = s(V(e))$, $e \in B$, where $s(V(e))$ is the support of the element $V(e) \in L_\Phi(B, m)$. It is clear that $\varphi(e) = \mathbf{0}$ if and only if $e = \mathbf{0}$. If $e, g \in B$ and $eg = \mathbf{0}$, then $V(e) \cdot V(g) = \mathbf{0}$ (see Corollary 2, thus $\varphi(e) \cdot \varphi(g) = \mathbf{0}$. Therefore,

$$\varphi(e \vee g) = s(V(e+g)) = s(V(e) + V(g)) = s(V(e)) + s(V(g)) = \varphi(e) + \varphi(g) = \varphi(e) \vee \varphi(g).$$

Using mathematical induction, we obtain that

$$\varphi\Big(\sup_{1\leqslant i\leqslant n} e_i\Big) = \sup_{1\leqslant i\leqslant n} \varphi(e_i)$$

for any finite set of pairwise disjoint idempotents $\{e_i\}_{i=1}^n \subset B$.

Let $\{e_i\}_{i=1}^\infty \subset B$ be a countable set of pairwise disjoint idempotents, and let $g_n = \sup_{1\leqslant i\leqslant n} e_i$, $n = 1, 2, \ldots$ Then $g_n \uparrow \sup_{n\geqslant 1} g_n = \sup_{i\geqslant 1} e_i := e$, and, by Proposition 6, we get $V(g_n) \uparrow V(e)$. Thus

$$\varphi(g_n) = s(V(g_n)) \uparrow s(V(e)) = \varphi(e), \quad \text{that is,} \quad \sup_{i\geqslant 1} \varphi(e_i) = \varphi(e).$$

Moreover,

$$\varphi(\mathbf{1}) = \varphi(e + Ce) = s(V(e + Ce)) = s(V(e) + V(Ce)) = s(V(e)) + s(V(Ce)) = \varphi(e) + \varphi(Ce),$$

that is, $\varphi(Ce) = \varphi(\mathbf{1}) - \varphi(e)$. Thus the mapping $\varphi : B \to B$ satisfies all the properties of a regular isomorphism from Definition 3.2.3 [8, Chapter III, § 3.2], so φ is an injective completely additive Boolean homomorphism [8, Chapter III, § 3.2, Remarks 3.2.4]. Using now Theorems 3 and 4 from [27], we get that there exists an injective normal homomorphism $T : L^0(B) \to L^0(B)$ such that $T(e) = \varphi(e)$ for all $e \in B$. In addition, the restriction $A = T|_{C(B)}$ is a $\|\cdot\|_\infty$-continuous injective homomorphism from $C(B)$ into $C(B)$.

If $e \in B$ then

$$\begin{aligned} V(e) &= V(\mathbf{1} - (\mathbf{1} - e))s(V(e)) = V(\mathbf{1})\varphi(e) - V(\mathbf{1} - e)\varphi(\mathbf{1} - e)\varphi(e) \\ &= V(\mathbf{1})\varphi(e) - V(\mathbf{1} - e)\varphi((\mathbf{1} - e)e) = V(\mathbf{1})\varphi(e), \end{aligned}$$

that is, $V(e) = V(\mathbf{1})\varphi(e)$. If

$$x = \sum_{i=1}^n \alpha_i e_i \in S(B) \subset C(B), \quad e_i \in B,\ e_ie_j = \mathbf{0},\ i \neq j,\ i, j = 1, \ldots, n,$$

is a step element then

$$V(x) = \sum_{i=1}^n \alpha_i V(e_i) = V(\mathbf{1}) \sum_{i=1}^n \alpha_i T(e_i) = V(\mathbf{1}) \cdot T(x).$$

Let $x \in C(B)$, and let $\{x_n\}_{n=1}^\infty \subset S(B)$ be a sequence of step elements such that $\|x_n - x\|_\infty \to 0$. Then

$$\big\|T(x_n) - T(x)\big\|_\infty = \big\|A(x_n) - A(x)\big\|_\infty \to 0.$$

Therefore,

$$\begin{aligned} \big\|V(x_n) - V(\mathbf{1})T(x)\big\|_\Phi &= \big\|V(\mathbf{1}) \cdot T(x_n) - V(\mathbf{1})T(x)\big\|_\Phi = \big\|V(\mathbf{1}) \cdot T(x_n - x)\big\|_\Phi \\ &\leqslant \big\|V(\mathbf{1})\big\|_\Phi \cdot \big\|T(x_n - x)\big\|_\infty \to \mathbf{0}. \end{aligned}$$

Since

$$\big\|V(x_n) - V(x)\big\|_\Phi = \big\||V(x_n - x)|\big\|_\Phi \leqslant \big\|V(|x_n - x|)\big\|_\Phi \leqslant \big\|x_n - x\big\|_\infty \big\|V(\mathbf{1})\big\|_\Phi \to \mathbf{0},$$

it follows that $V(x) = V(\mathbf{1})T(x)$.

Let us show that the equality $Vx = V(\mathbf{1}) \cdot T(x)$ holds for all $x \in L_\Phi(B,m)$. It suffices to check this equality for all $0 \leqslant x \in L_\Phi(B,m)$. Let $0 \leqslant x \in L_\Phi(B,m)$, and let $0 \leqslant x_n = x \cdot \chi_{\{0 \leqslant x \leqslant n\}} \in C(B)$. Since $x_n \uparrow x$ and the norm $\|\cdot\|_\Phi$ is order continuous norm (Proposition 6), it follows that

$$\big\|V(\mathbf{1}) \cdot T(x_n) - V(x)\big\|_\Phi = \big\|V(x_n) - V(x)\big\|_\Phi = \big\|x_n - x\big\|_\Phi \to 0.$$

Using now the convergence $T(x_n) \uparrow T(x)$, we obtain that $V(x) = V(\mathbf{1}) \cdot T(x)$, where $\mathbf{0} \leqslant V(\mathbf{1}) \in L_\Phi(B,m)$. $\rhd$

In the case of the Orlicz function $\Phi(t) = t^p/p$, Theorem 1 entails the following description of all positive linear isometries acting in a Banach–Kantorovich L^p-space.

Corollary 3. *Let m be the Maharam measure on a complete Boolean algebra B, and let $V : L^p(B,m) \to L^p(B,m)$, $1 \leqslant p < \infty$, be a positive linear isometry. Then there exists an injective normal homomorphism $T : L^0(B) \to L^0(B)$ and a positive element $y \in L^p(B,m)$ such that $V(x) = y \cdot T(x)$ for all $x \in L^p(B,m)$.*

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Botir S. Zakirov
Tashkent State Transport University,
1 Temiryulchilar St., Tashkent 100167, Uzbekistan,
Doctor of Physical and Mathematical Sciences, Professor
E-mail: botirzakirov@list.ru
https://orcid.org/0000-0001-8381-8518

Vladimir I. Chilin
National University of Uzbekistan,
Vuzgorodok, Tashkent 100174, Uzbekistan,
Doctor of Physical and Mathematical Sciences, Professor
E-mail: vladimirchil@gmail.com
https://orcid.org/0000-0002-7936-9649

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ПОЛОЖИТЕЛЬНЫЕ ИЗОМЕТРИИ ПРОСТРАНСТВ ОРЛИЧА — КАНТОРОВИЧА

Закиров Б. С.[1], Чилин В. И.[2]

[1] Ташкентский государственный транспортный университет,
Узбекистан, 100167, Ташкент, ул. Темирйулчилар, 1;
[2] Национальный университет Узбекистана,
Узбекистан, Ташкент, 100174, Вузгородок

E-mail: botirzakirov@list.ru, vladimirchil@gmail.com

Аннотация. Пусть B полная булева алгебра, $Q(B)$ стоуновский компакт для B, и пусть $C_\infty(Q(B))$ коммутативная алгебра всех непрерывных функций $x : Q(B) \to [-\infty, +\infty]$, принимающихо, значения $\pm\infty$ на нигде не плотных подмножествах из $Q(B)$. Мы рассматриваем пространства Орлича — Канторовича $(L_\Phi(B,m), \|\cdot\|_\Phi) \subset C_\infty(Q(B))$ с нормой Люксембурга, построенные по функции Орлича Φ и векторнозначной мере m со значениями в алгебре действительных измеримых функций. Показывается, что в случае наличия (Δ_2)-условия для функции Орлича Φ, норма $\|\cdot\|_\Phi$ является порядково непрерывной, т. е. $\|x_n\|_\Phi \downarrow \mathbf{0}$ для любой последовательности $\{x_n\} \subset L_\Phi(B,m)$, $x_n \downarrow \mathbf{0}$. Кроме того, в этом случае, норма $\|\cdot\|_\Phi$ является строго монотонной, т. е. из $|x| \lneqq |y|$ $x,y \in L_\Phi(B,m)$ следует, что $\|x\|_\Phi \lneqq \|y\|_\Phi$. При этом, для положительных элементов $x,y \in L_\Phi(B,m)$ равенство $\|x+y\|_\Phi = \|x-y\|_\Phi$ выполняется тогда и только тогда, когда $x \cdot y = 0$. Используя эти свойства нормы Люксембурга, доказывается, что для любой положительной линейной изометрии $V : L_\Phi(B,m) \to L_\Phi(B,m)$ существуют такие инъективный нормальный гомоморфизм $T : C_\infty(Q(B)) \to C_\infty(Q(B))$ и положительный элемент $y \in L_\Phi(B,m)$, что $V(x) = y \cdot T(x)$ для всех $x \in L_\Phi(B,m)$.

Ключевые слова: пространство Банаха — Канторовича, функция Орлича, векторнозначная мера, положительная изометрия, нормальный гомоморфизм.

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